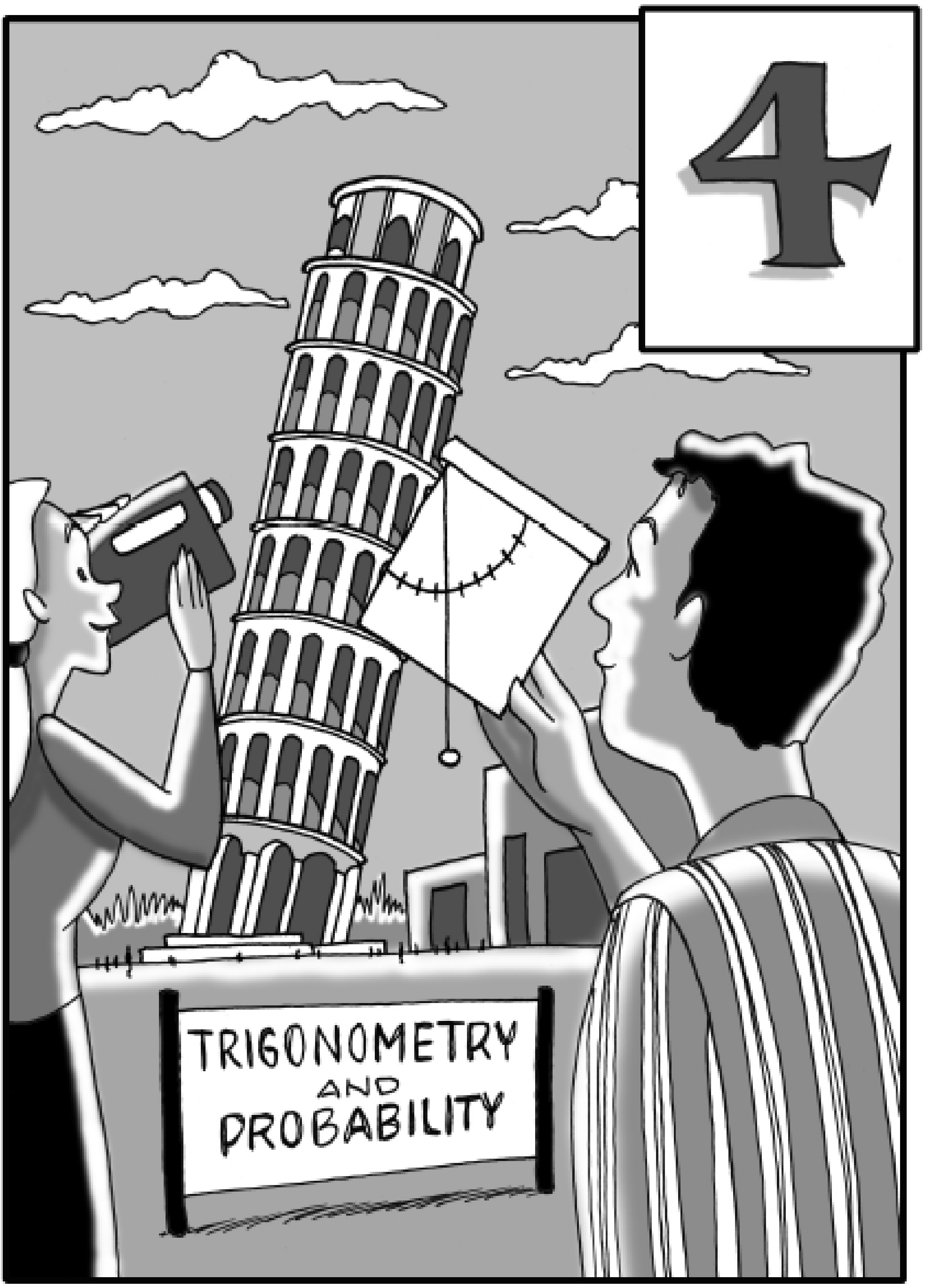

4
TRIGONOMETRY
AND
PROBABILITY

CHAPTER 4 Trigonometry and Probability

In Chapter 3, you investigated similarity and discovered that similar triangles have special relationships. In this chapter, you will discover that the side ratios in a right triangle can serve as a powerful mathematical tool that allows you to find missing side lengths and missing angle measures for any right triangle. You will also learn how these ratios (called trigonometric ratios) can be used in solving problems.

You will also develop additional prediction skills as you extend your understanding of probability. You will examine different models to represent possibilities and to assist you in calculating probabilities.

Guiding Question

Mathematically proficient students will use appropriate tools strategically.

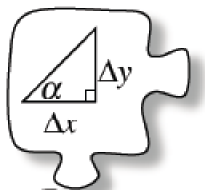
As you work through this chapter, ask yourself:

Can I use the available tools to solve problems and decide which tool might be the most helpful?

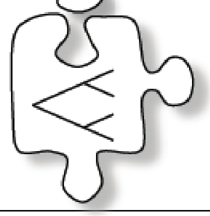
In this chapter, you will learn:

- The trigonometric ratio of tangent.
- How the tangent ratio is connected to the slope of a line.
- How to apply trigonometric ratios to find missing measurements in right triangles.
- How to model real world situations with right triangles and use trigonometric ratios to solve problems.
- Several ways to model probability situations, such as tree diagrams and area models.
- How to formalize methods for computing probabilities of unions, intersections, and complements of events.
- How to find expected value in games of chance.

Chapter Outline



Section 4.1 Students will investigate the relationship between the slope of a line and the slope angle. The slope ratio will be used to find missing measurements of a right triangle and to solve real world problems.



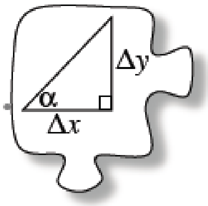
Section 4.2 Students will continue their study of probability by using tree diagrams and area models to calculate probabilities of events that are not equally likely. Students will calculate expected values and probabilities of unions, intersections, and complements of events.

Chapter 4 Teacher Guide

Section	Lesson	Days	Lesson Objectives	Materials	Homework
4.1	4.1.1	1	Constant Ratios in Right Triangles	• Protractors • Lesson 4.1.1 Res. Pgs.	4-6 to 4-11
	4.1.2	1	Connecting Slope Ratios to Specific Angles	• Lesson 4.1.2 Res. Pg. • Tracing Paper • Student work from Lesson 4.1.1	4-17 to 4-22
	4.1.3	1	Expanding the Trig Table	• Computer lab or computer with projector OR Lesson 4.1.3 Res. Pg. • Lesson 4.1.2 Res. Pg.	4-27 to 4-32
	4.1.4	1	The Tangent Ratio	• Lesson 4.1.2 Res. Pg. OR • Lesson 4.1.3 Res. Pg. • Tracing paper	4-39 to 4-44
	4.1.5	1	Applying the Tangent Ratio	• Lesson 4.1.5A Res. Pg. • Meter sticks or tape measures • Clinometers (made with straw, string, cardboard, glue, tape, ¼" washers and the Lesson 4.1.5B and 4.1.5C Res. Pgs.)	4-47 to 4-52
4.2	4.2.1	1	Using an Area Model	None	4-58 to 4-63
	4.2.2	1	Using a Tree Diagram	None	4-69 to 4-74
	4.2.3	1	Probability Models	None	4-81 to 4-86
	4.2.4	1	Unions, Intersections, and Complements	• Lesson 4.2.4A-B Res. Pgs.	4-95 to 4-100
	4.2.5	2	Expected Value	• Lesson 4.2.5 Res. Pg. • Number cube	4-110 to 4-115 and 4-116 to 4-121
Chapter Closure		Various Options			

Total: 11 days plus time for Chapter Closure and Assessment

4.1.1 What patterns can I use?

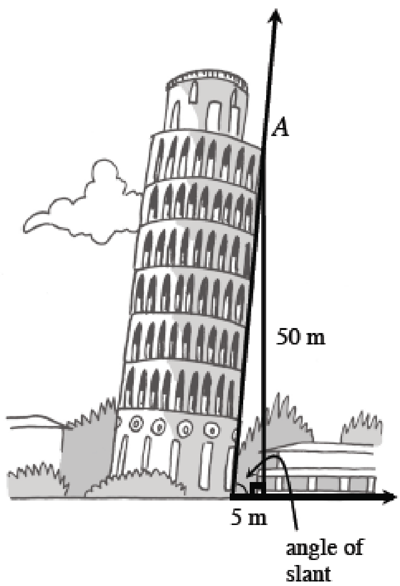


Constant Ratios in Right Triangles

In Chapter 3, you looked for relationships between triangles and ways to determine if they are similar or congruent. Now you are going to focus your attention on slope triangles, which were used in algebra to describe linear change. Are there geometric patterns within slope triangles themselves that you can use to answer other questions? In this lesson, you will look closely at slope triangles on different lines to explore their patterns.

4-1. LEANING TOWER OF PISA

For centuries, people have marveled at the Leaning Tower of Pisa due to its slant and beauty. Ever since construction of the tower started in the 1100s, the tower has slowly tilted south and has increasingly been at risk of falling over. It is feared that if the angle of slant ever falls below 83°, the tower will collapse.



Engineers closely monitor the angle at which the tower leans. With careful measuring, they know that the point labeled *A* in the diagram at right is now 50 meters off the ground. Also, they determined that when a weight is dropped from point *A*, it lands 5 meters from the base of the tower, as shown in the diagram.

a. With the measurements provided above, what can you determine?

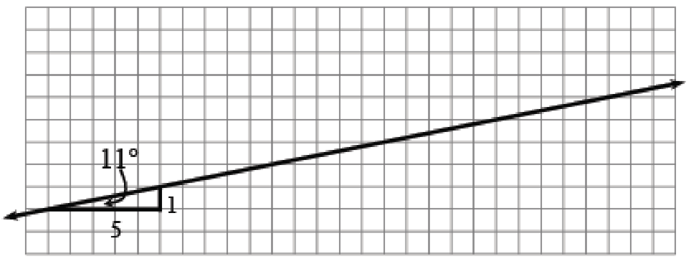
b. Can you determine the angle at which the tower leans? Why or why not?

c. At the end of Section 4.1, you will know how to find the angle for this situation and many others. However, at this point, how else can you describe the "lean" of the leaning tower?

4-2. PATTERNS IN SLOPE TRIANGLES

In order to find an angle (such as the angle at which the Leaning Tower of Pisa leans), you need to investigate the relationship between the angles and the sides of a right triangle. You will start by studying slope triangles. Obtain the Lesson 4.1.1 Resource Pages from your teacher and find the graph shown below. Notice that one slope triangle has been drawn for you. Note: For the next several lessons angle measures will be rounded to the nearest degree.

a. Draw three new slope triangles on the line. Each should be a different size. Label each triangle with as much information as you can, such as its horizontal and vertical lengths and its angle measures.



b. Explain why all of the slope triangles on this line must be similar.

c. Since the triangles are similar, what does that tell you about the slope ratios?

d. Confirm your conclusion by writing the slope ratio for each triangle as a fraction, such as $\frac{\Delta y}{\Delta x}$. (Note: Δy represents the vertical change or "rise," while Δx represents the horizontal change or "run.") Then change the slope ratio into decimal form and compare.

4-3. Tara thinks she sees a pattern in these slope triangles, so she decides to make some changes in order to investigate whether or not the patterns remain true.

a. She asks, "*What if I draw a slope triangle on this line with $\Delta y = 6$? What would be the Δx of my triangle?*" Answer her question and explain how you figured it out.

b. "*What if Δx is 40?*" she wonders. "*Then what is Δy?*" Find Δy, and explain your reasoning.

c. Tara wonders, "*What if I draw a slope triangle on a different line? Can I still use the same ratio to find a missing Δx- or Δy-value?*" Discuss this question with your team and explain to Tara what she could expect.

4-4. CHANGING LINES

In part (c) of problem 4-3, Tara asked, "*What if I draw my triangle on a different line?*" With your team, investigate what happens to the slope ratio and slope angle when the line is different. Use the grids provided on your Lesson 4.1.1 Resource Pages to graph the lines described below. Use the graphs and your answers to the questions below to respond to Tara's question.

a. On graph A, graph the line $y = \frac{2}{5}x$. What is the slope ratio for this line? What does the slope angle appear to be? Does the information about this line support or change your conclusion from part (c) of problem 4-3? Explain.

b. On graph B, you are going to create $\angle QPR$ so that it measures 18°. First, place your protractor so that point P is the vertex. Then find 18° and mark and label a new point, R. Draw ray $\overrightarrow{PR}$ to form $\angle QPR$. Find an approximate slope ratio for this line.

c. Graph the line $y = x + 4$ on graph C. Draw a slope triangle and label its horizontal and vertical lengths. What is $\frac{\Delta y}{\Delta x}$ (the slope ratio)? What is the slope angle?

4-5. TESTING CONJECTURES

The students in Ms. Coyner's class are writing conjectures based on their work today. As a team, decide if you agree or disagree with each of the conjectures below. Explain your reasoning.

- All slope triangles have a ratio $\frac{1}{5}$.
- If the slope ratio is $\frac{1}{5}$, then the slope angle is approximately $11°$.
- If the line has an $11°$ slope angle, then the slope ratio is approximately $\frac{1}{5}$.
- Different lines will have different slope angles and different slope ratios.

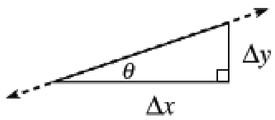
MATH NOTES

METHODS AND MEANINGS

Slope and Angle Notation

The slope of a line is the ratio of the vertical distance to the horizontal distance in a slope triangle formed by two points on a line. The vertical part of the triangle is called Δy, (read "change in *y*"), while the horizontal part of the triangle is called Δx (read "change in *x*"). Slope can then be written as $\frac{\Delta y}{\Delta x}$. Slope indicates both how steep the line is and its direction, upward or downward, left to right.

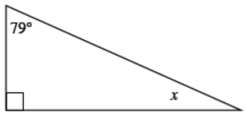
When a side length in a triangle is missing, that length is often assigned a variable from the English alphabet such as *x*, *y*, or *z*. However, sometimes you need to distinguish between an unknown side length and an unknown angle measure. With that in mind, mathematicians sometimes use Greek letters as variables for angle measurement. The most common variable for an angle is the Greek letter θ (*theta*), pronounced "THAY-tah." Two other Greek letters commonly used include α (*alpha*), and β (*beta*), pronounced "BAY-tah."

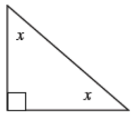


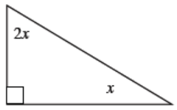
When a right triangle is oriented like a slope triangle, such as the one in the diagram above, the angle the line makes with the horizontal side of the triangle is called a **slope angle**.

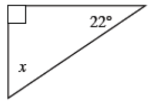
Review & Preview

4-6. Use what you know about the angles of a triangle to find the value of x and the angles in each triangle below.

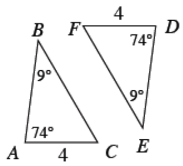
a.


b.


c.


d.


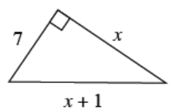
4-7. Use the triangles at right to answer the following questions.



a. Are the triangles at right similar? How do you know? Show your reasoning in a flowchart.

b. Examine your work from part (a). Are the triangles also congruent? Explain why or why not.

4-8. As Randi started to solve for x in the diagram at right, she wrote the equation $7^2 + x^2 = (x+1)^2$.



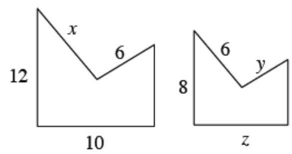
a. Is Randi's equation valid? Explain your thinking.

b. To solve her equation, first rewrite $(x+1)^2$ by multiplying $(x+1)(x+1)$. You may want to review the Math Notes box in Lesson 2.2.2.

c. Now solve your equation for x.

d. What is the perimeter of Randi's triangle?

4-9. Assume that the shapes at right are similar. Find the values of x, y, and z.



4-10. ROLL AND WIN

You begin the game *Roll and Win* by picking a number. Then you roll two regular dice, each numbered 1 through 6, and *add* the numbers that come up together. If the sum is the number you chose, you win a point. For example, if you choose "11," and a 6 and a 5 are rolled, you win!

a. What is the sample space, which can be thought of as the set of all the possible outcomes, when two dice are rolled and their numbers added?

b. One way to analyze this situation is to make a model of all the possible outcomes like the one at right. Copy and complete this table of sums on your paper. Are each of the outcomes in this table equally likely?

Dice #1

Dice #2	1	2	3	4	5	6
1				**5**		
2		**4**				
3						
4			**7**			
5						
6						

c. What is P(even)? P(10)? P(15)?

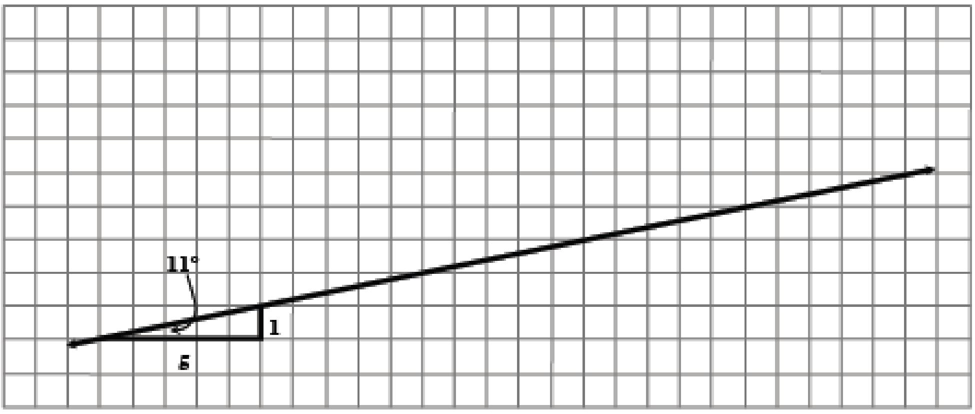
d. Which sum is the most likely result? What is the probability of rolling that sum?

4-11. The temperature in San Antonio, Texas is currently 77°F and is increasing by 3° per hour. The current temperature in Bombay, India is 92°F and the temperature is dropping by 2° per hour. When will it be as hot in San Antonio as it is in Bombay? What will the temperature be?

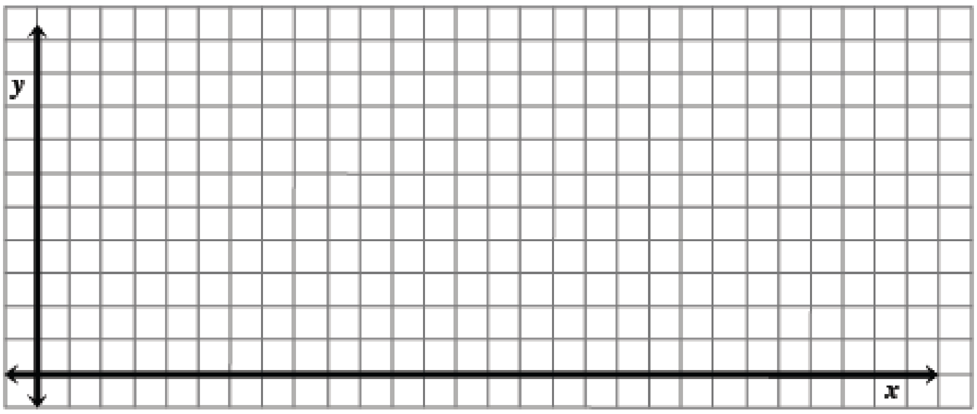
Lesson 4.1.1 Resource Page Page 1 of 2

Patterns In Slope Triangles

Problem 4-2 part (a) and problem 4-3.



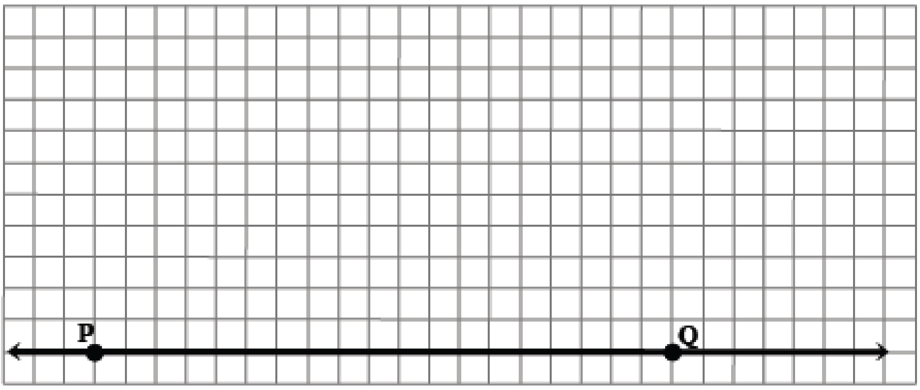
Graph A: (part (a) of problem 4-4)



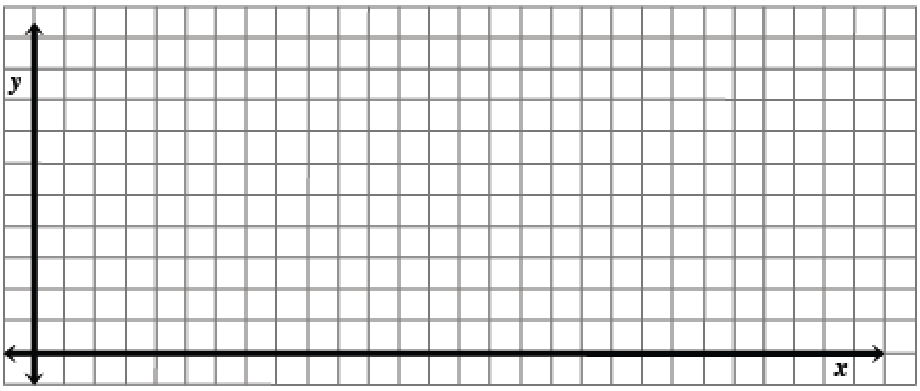
Lesson 4.1.1 Resource Page Page 2 of 2

Patterns In Slope Triangles

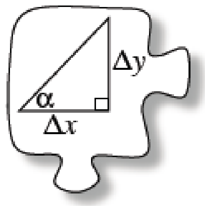
Graph B: (part (b) of problem 4-4)



Graph C: (part (c) of problem 4-4)



4.1.2 How important is the angle?



Connecting Slope Ratios to Specific Angles

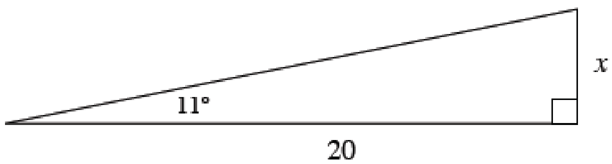
In Lesson 4.1.1, you started **trigonometry**, the study of the measures of triangles. As you continue to investigate right triangles with your team today, use the following questions to guide your discussion:

What do I know about this triangle?

How does this triangle relate to other triangles?

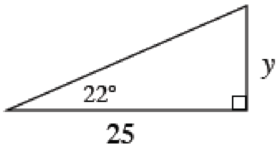
Which part is Δx? Which part is Δy?

4-12. What do you know about this triangle? To what other triangles does it relate? Use any information you have to solve for x.

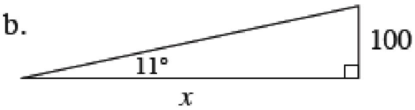


4-13. For each triangle below, find the missing angle or side length. Use your work from Lesson 4.1.1 to help you.

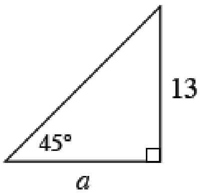
a.



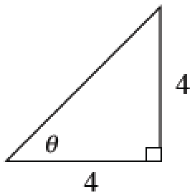
b.



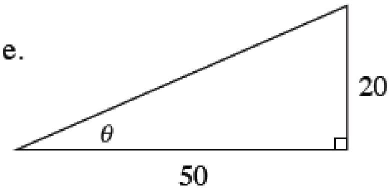
c.



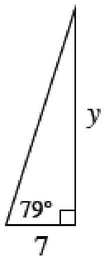
d.



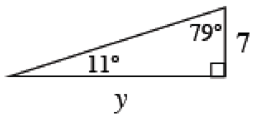
e.



f.



4-14. Sheila says the triangle in part (f) of problem 4-13 is the same as her drawing at right.



a. Do you agree? Use tracing paper to convince yourself of your conclusion.

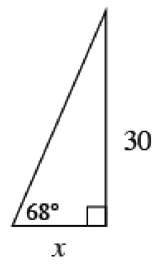
b. Use what you know about the slope ratio of 11° to find the slope ratio for 79°.

c. What is the relationship of 11° and 79°? What is the relationship between their slope ratios?

4-15. For what other angles can you find the slope ratios based on the work you did in Lesson 4.1.1?

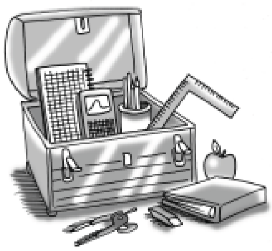
a. For example, since you know the slope ratio for 22°, what other angle do you know the slope ratio for? Use tracing paper to find a slope ratio for the complement of each slope angle you know. Use tracing paper to help re-orient the triangle if necessary.

b. Use this information to find x in the diagram at right.



c. Write a conjecture about the relationship of the slope ratios for complementary angles. You may want to start with, "*If one angle of a right triangle has the slope ratio* $\frac{a}{b}$, *then …*"

4-16. BUILDING A TRIGONOMETRY TABLE TOOLKIT



So far you have looked at several similar slope triangles and their corresponding slope ratios. These relationships will be very useful for finding missing side lengths or angle measures of right triangles for the rest of this chapter.

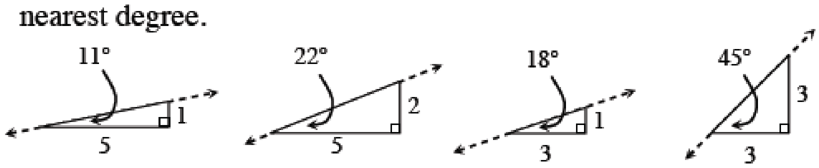
Before you forget this valuable information, organize information about the triangles and ratios you have discovered so far in the table on the Lesson 4.1.2 ("Trig Table Toolkit") Resource Page provided by your teacher. Keep it in a safe place for future reference. Include all of the angles you have studied up to this point. An example for 11° is filled in on the table to get you started.

MATH NOTES

METHODS AND MEANINGS

Slope Ratios and Angles

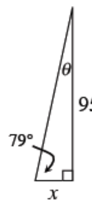
In Lesson 4.1.1, you discovered that certain slope angles produce slope triangles with special ratios. Below are the triangles you have studied so far. Note that the angles below are rounded to the nearest degree.



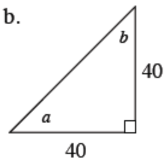
Review & Preview

4-17. Use your Trig Table Toolkit from problem 4-16 to help you find the value of each variable below.

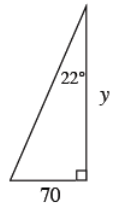
a.



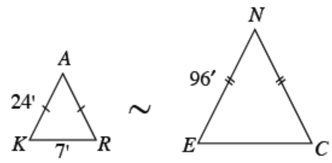
b.



c.



4-18. The triangles shown at right are similar.



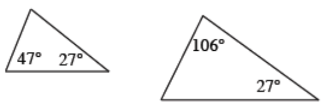
a. What is the ratio of side length NE to side length AK?

b. Use a ratio to compare the perimeters of ΔENC and ΔKAR. How is the perimeter ratio related to the side length ratio?

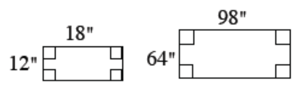
c. If you have not already done so, find the length of $\overline{EC}$.

4-19. Examine each pair of figures below. Are they similar? Explain how you know.

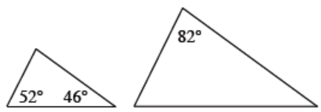
a.



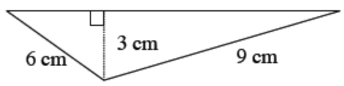
b.



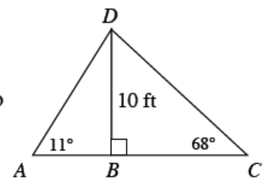
c.



4-20. Find the area and perimeter of the triangle at right.



4-21. Examine the figure at right, which is not drawn to scale. Which is longer, $\overline{AB}$ or $\overline{BC}$? Explain your answer.



4-22. Joan and Jim are planning a dinner menu including a main dish and dessert. They have 4 main dish choices (steak, vegetable-cheese casserole, turkey burgers, and vegetarian lasagna) and 3 dessert choices (chocolate brownies, strawberry ice cream, and chocolate chip cookies.)

a. Joan and Jim would like to know how many different dinner menus they have to choose from. One way to make sure you have considered the entire sample space – all the possible menu outcomes – is to make a table like the one at right. How many different menus are there?

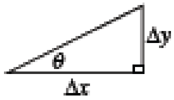
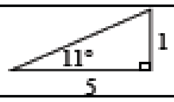
	steak	vegetable casserole	turkey burgers	vegetable lasagna
chocolate brownies				
strawberry ice cream				
chocolate chip cookies				

b. Assume the main dish choice and the dessert choice are both chosen randomly. Are all the menus equally likely?

c. What is the probability they pick a menu without meat? What is the probability they pick a menu with chocolate?

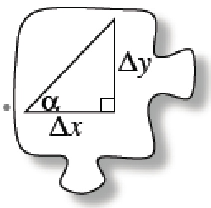
Lesson 4.1.2 Resource Page

Trig Table Toolkit

Angle	Slope triangle	Approximate slope ratio as a fraction and a decimal
$\theta°$		$\frac{\Delta y}{\Delta x}$
0°		
8°		
11°		$\frac{1}{5} = 0.2$
18°		
22°		
45°		
55°		
68°		
70°		
72°		
79°		
83°		
84°		
89°		

Note: Angle measures are rounded to the nearest degree.

4.1.3 What if the angle changes?



Expanding the Trig Table

In the last few lessons, you found the slope ratios for several angles. However, so far you are limited to using the slope angles that are currently in your Trig Table. How can you find the ratios for other angles? And how are the angles related to the ratio?

Today your goal is to determine ratios for more angles and to find patterns. As you work today, keep the following questions in mind:

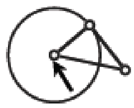
What happens to the slope ratio when the angle increases? Decreases?

What happens to the slope ratio when the angle is 0°? 90°?

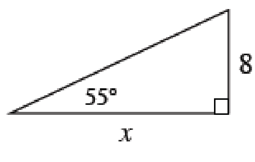
When is a slope ratio more than 1? When is it less than 1?

4-23. On your paper, draw a slope triangle with a slope angle of 45°.

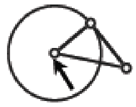
a. Now visualize what would happen to the triangle if the slope angle increased to 55°. Which would be longer? Δy or Δx? Explain your reasoning.



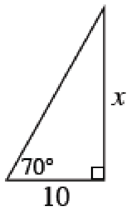
b. Using your dynamic geometry tool (or the Lesson 4.1.3 Resource Page), create a triangle with a slope angle measuring 55°. Then use the resulting slope ratio to solve for x in the triangle at right. (Note: The triangle at right is not drawn to scale.)



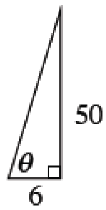
4-24. Copy each of the following triangles onto your paper. Decide whether or not the given measurements are possible. If the triangle is possible, find the value of x or θ. Use the dynamic geometry tool to find the appropriate slope angles or ratios needed. If technology is not available, your teacher will provide a Lesson 4.1.3 Resource Page with the needed ratios. Round angle measures to the nearest degree. If a triangle's indicated measurement is not possible, explain why.



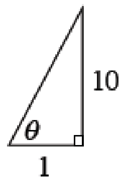
a.



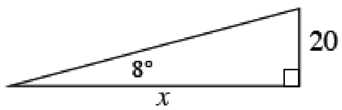
b.



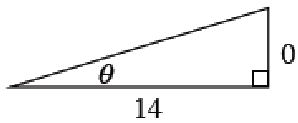
c.



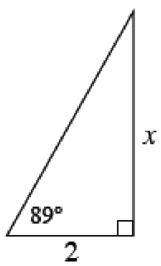
d.



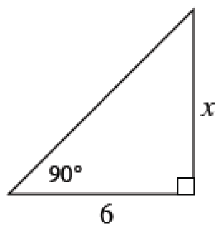
e.



f.



g.



4-25. If you have not already, add these new slope ratios with their corresponding angles to your Trig Table Toolkit. Be sure to draw and label the triangle for each new angle. Summarize your findings—which slope triangles did not work? Do you see any patterns?

4-26. What statements can you make about the connections between slope angle and slope ratio? In your Learning Log, write down all of your observations from this lesson. Be sure to answer the questions given at the beginning of the lesson (reprinted below). Title this entry, "Slope Angles and Slope Ratios" and include today's date.

What happens to the slope ratio when the angle increases? Decreases?

What happens to the slope ratio when the angle is 0°? 90°?

When is a slope ratio more than 1? When is it less than 1?

METHODS AND MEANINGS

MATH NOTES

Sequences

A sequence is a function in which the independent variable is a positive integer (usually called the "term number") and the dependent value is the term value. A sequence is usually written as a list of numbers.

Arithmetic Sequences

In an arithmetic sequence, the **common difference** between terms is constant. For example, in the arithmetic sequence 4, 7, 10, 13, …, the common difference is 3.

The equation for an arithmetic sequence is: $t(n) = mn + b$ or $a_n = mn + a_0$ where n is the term number, m is the common difference, and b or a_0 is the zeroth term. Compare these equations to a continuous linear function $f(x) = mx + b$ where m is the growth (slope) and b is the starting value (y-intercept).

For example, the arithmetic sequence 4, 7, 10, 13, … could be represented by $t(n) = 3n + 1$ or by $a_n = 3n + 1$. (Note that "4" is the first term of this sequence, so "1" is the zeroth term.)

Another way to write the equation of an arithmetic sequence is by using the first term in the equation, as in $a_n = m(n-1) + a_1$, where a_1 is the first term. The sequence in the example could be represented by $a_n = 3(n-1) + 4$.

You could even write an equation using any other term in the sequence. The equation using the fourth term in the example would be $a_n = 3(n-4) + 13$.

Geometric Sequences

In a geometric sequence, the **common ratio** or **multiplier** between terms is constant. For example, in the geometric sequence 6, 18, 54, …, the multiplier is 3. In the geometric sequence 32, 8, 2, $\frac{1}{2}$, …, the common multiplier is $\frac{1}{4}$.

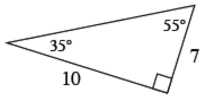
The equation for a geometric sequence is: $t(n) = ab^n$ or $a_n = a_0 \cdot b^n$ where n is the term number, b is the sequence generator (the multiplier or common ratio), and a or a_0 is the zeroth term. Compare these equations to a continuous exponential function $f(x) = ab^x$ where b is the growth (multiplier) and a is the starting value (y-intercept).

For example, the geometric sequence 6, 18, 54, … could be represented by $t(n) = 2 \cdot 3^n$ or by $a_n = 2 \cdot 3^n$.

You can write a first term form of the equation for a geometric sequence as well: $a_n = a_1 \cdot b^{n-1}$. For the example, first term form would be $a_n = 6 \cdot 3^{n-1}$.

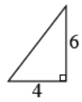
Review & Preview

4-27. Ben thinks that the slope ratio for this triangle is $\frac{7}{10}$. Carlissa thinks the ratio is $\frac{10}{7}$. Who is correct? Explain your thinking fully.

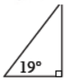


4-28. Use your observations from problem 4-26 to answer the following questions:

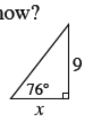
a. Thalia did not have a tool to help her find the slope angle in the triangle at right. However, she claims that the slope angle has to be more than 45°. Do you agree with Thalia? Why?



b. Lyra was trying to find the slope ratio for the triangle at right, and she says the answer is $\frac{\Delta y}{\Delta x} = 2.675$. Isiah claims that cannot be correct. Who is right? How do you know?



c. Without finding the actual value, what information do you know about x in the diagram at right?



4-29. Examine each sequence below. State whether it is arithmetic, geometric, or neither. For the sequences that are arithmetic or geometric, find the equation for $t(n)$ or a_n. Refer to the Math Notes box in this lesson if you need additional help.

a. 1, 4, 7, 10, 13, …

b. 0, 5, 12, 21, 32, …

c. 2, 4, 8, 16, 32, …

d. 5, 12, 19, 26, …

4-30. Edwina has created her own Shape Bucket and has provided the clues below about her shapes. List one possible group of shapes that could be in her bucket.

$$P(\text{equilateral}) = 1$$

$$P(\text{triangle}) = \frac{1}{3}$$

4-31. Renae has programmed her music player to play all five songs in her playlist in a random order without repeating songs.

PLAYLIST

a. **I Love My Mama** (country) by the Strings of Heaven

b. **Don't Call Me Mama** (country) Duet by Sapphire and Hank Tumbleweed

c. **Carefree and Blue** (R & B) by Sapphire and Prism Escape

d. **Go Back To Mama** (Rock) Duet by Bjorn Free and Sapphire

e. **Smashing Lollipops** (Rock) by Sapphire

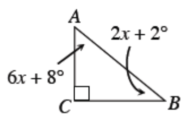
a. What is the probability that the first song is a country song?

b. If the first song is a country song, does that affect the probability that the second song is a country song? Explain your thinking.

c. As songs are playing, the number of songs left to play decreases. Therefore, the probability of playing each of the remaining songs depends on which songs that have played before it. This is an example of events that are **not independent**. If Renae has already listened to "Don't Call Me Mama," "Carefree and Blue," and "Smashing Lollipops," what is the probability that one of the singers of the fourth song will be Sapphire? Explain your reasoning.

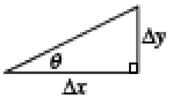
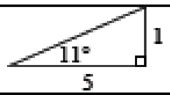
d. To get home, Renae can take one of four buses: #41, #28, #55, or #81. Once she is on a bus, she will randomly select one of the following equally likely activities: listening to her music player, writing a letter, or reading a book. Her choice of bus and choice of entertainment are **independent events**, because the bus that Renae took did not affect which activity she chose. For example, what is the probability that Renae writes a letter if she takes the #41 bus? What is the probability that Renae writes a letter if she takes the #55 bus?

4-32. Use what you know about the sum of the angles of a triangle to find $m\angle ABC$ and $m\angle BAC$. Are these angles acute or obtuse? Find the sum of these two angles. How can you describe the relationship of these two angles?

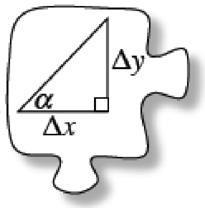


Lesson 4.1.3 Resource Page

Trig Table Toolkit

Angle	Slope triangle	Approximate slope ratio as a fraction and a decimal
$\theta°$		$\frac{\Delta y}{\Delta x}$
0°		0
8°		$\frac{52}{370} = \frac{26}{185} \approx 0.141$
11°		$\frac{1}{5} = 0.2$
18°		$\frac{1}{3} \approx 0.33$
22°		$\frac{2}{5} = 0.4$
45°		$\frac{1}{1} = 1$
55°		$\frac{10}{7} \approx 1.429$
68°		$\frac{5}{2} = 2.5$
70°		$\frac{13.737}{5} \approx 2.747$
72°		$\frac{3}{1} = 3$
79°		$\frac{5}{1} = 5$
83°		$\frac{50}{6} = \frac{25}{3} \approx 8.33$
84°		$\frac{10}{1} = 10$
89°		$\frac{5729}{100} \approx 57.29$

4.1.4 What about other right triangles?

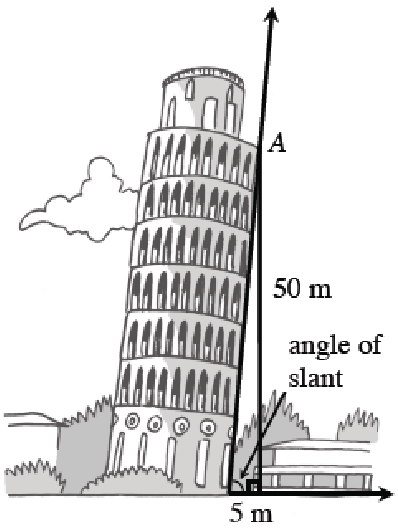


The Tangent Ratio

In Lesson 4.1.2 you started a Trig Table Toolkit of angles and their related slope ratios. Unfortunately, you only have information for a few angles. How can you quickly find the ratios for other angles when a computer is not available or when an angle is not on your Trig Table? Do you have to draw each angle to get its slope ratio? Or is there another way?

4-33. WILL IT TOPPLE?

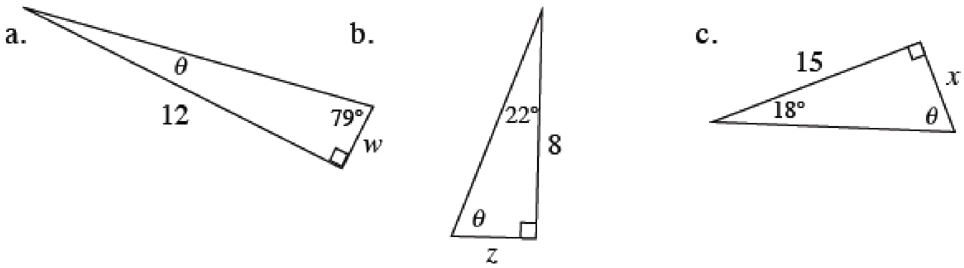
In problem 4-1, you learned that the Leaning Tower of Pisa is expected to collapse once its angle of slant is less than 83°. Currently, the top of the seventh story (point A in the diagram at right) is 50 meters above the ground. In addition, when a weight is dropped from point A, it lands 5 meters from the base of the tower, as shown in the diagram.



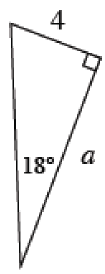
a. What is the slope ratio for the tower?

b. Use your Trig Table Toolkit to determine the angle at which the Leaning Tower of Pisa slants. Is it in immediate danger of collapse?

4-34. Solve for the variables in the triangles below. It may be helpful to first orient the triangle (by rotating your paper or by using tracing paper) so that the triangle resembles a slope triangle. Use your Trig Table for reference.



4-35. MULTIPLE METHODS



Tiana, Mae Lin, Eddie, and Amy are looking at the triangle at right and trying to find the missing side length.

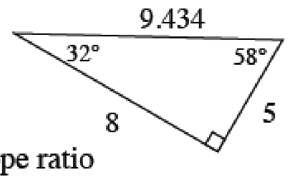
a. Tiana declares, "*Hey! We can rotate the triangle so that 18° looks like a slope angle, and then $\Delta y = 4$.*" Will her method work? If so, use her method to solve for a. If not, explain why not.

b. Mae Lin says, "*I see it differently. I can tell $\Delta y = 4$ without turning the triangle.*" How can she tell? Explain one way she could know.

c. Eddie replies, "*What if we use 72° as our slope angle? Then $\Delta x = 4$.*" What is he talking about? Discuss with your team and explain using pictures and words.

d. Use Eddie's observation in part (c) to confirm your answer to part (a).

4-36. USING A SCIENTIFIC CALCULATOR



Examine the triangle at right.

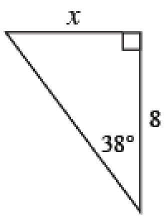
a. According to the triangle at right, what is the slope ratio for 32°? Explain how you decided to set up the ratio. Write the ratio in both fraction and decimal form.

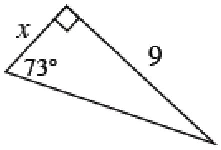
b. What is the slope ratio for the 58° angle? How do you know?

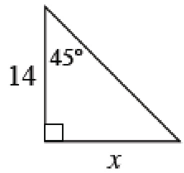
c. Scientific calculators have a button that will give the slope ratio when the slope angle is entered. In part (a), you calculated the slope ratio for 32° as 0.625. Use the "tan" button on your calculator to verify that you get ≈ 0.625 when you enter 32°. Does that button give you ≈ 1.600 when you enter 58°? Be ready to help your teammates find and use the button on their calculator.

d. The ratio in a right triangle that you have been studying is referred to as the **tangent ratio**. When you want to find the slope ratio of an angle, such as 32°, it is written "$\tan 32°$." So, an equation for this triangle can be written as $\tan 32° = \frac{5}{8}$. Read more about the tangent ratio in the Math Notes box for this lesson.

4-37. For each triangle below, trace the triangle on tracing paper. Label its legs Δy and Δx based on the given slope angle. Then write an equation (such as $\tan 14° = \frac{x}{5}$), use your scientific calculator to find a slope ratio for the given angle, and solve for x.

a.


b.


c.


4-38. How do you set up a tangent ratio equation? How do you know which side of the triangle is Δy? How can you use your scientific calculator to find a slope ratio? Write a Learning Log entry about what you learned today. Be sure to include examples or refer to your work from today. Title this entry "The Tangent Ratio" and include today's date.

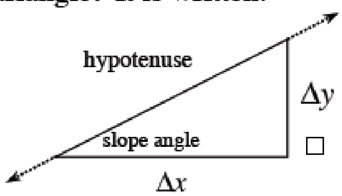
MATH NOTES

METHODS AND MEANINGS

The Tangent Ratio

For any slope angle in a slope triangle, the ratio that compares the Δy to Δx is called the **tangent ratio**. The ratio for any angle is constant, regardless of the size of the triangle. It is written:

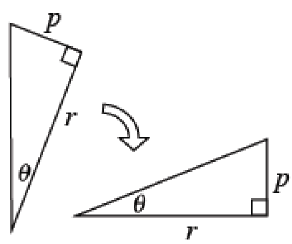
$$\tan(\text{slope angle}) = \frac{\Delta y}{\Delta x}$$



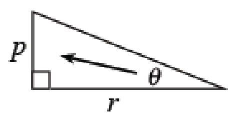
One way to identify which side is Δy and Δx is to first reorient the triangle so that it looks like a slope triangle, as shown at right.

For example, when the triangle at right is rotated, the resulting slope triangle helps to show that the tangent of θ is $\frac{p}{r}$, since θ is the slope angle, p is Δy and r is Δx. This is written:

$$\tan\,\theta = \frac{p}{r}$$



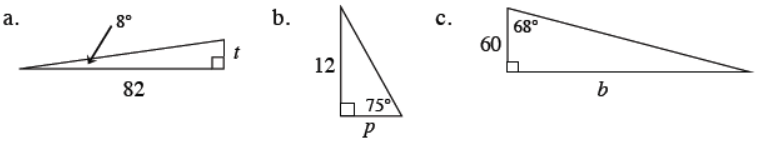
Whether the triangle is oriented as a slope triangle or not, you can identify Δy as the leg that is always opposite (across the triangle from) the angle, while Δx is the leg closest to the angle.



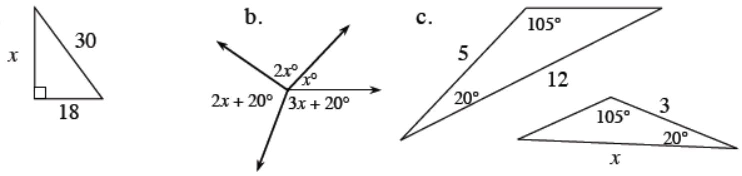
$$\tan\,\theta = \frac{\text{opposite leg}}{\text{adjacent leg}} = \frac{p}{r}$$

Review & Preview

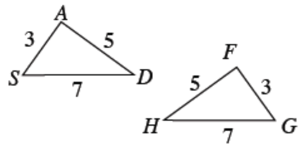
4-39. Find the missing side length for each triangle. Use the tangent button on your calculator to help.



4-40. Use the relationships in the diagrams below to write an equation and solve for x.



4-41. What is the relationship of the triangles at right? Justify your conclusion using rigid transformations.



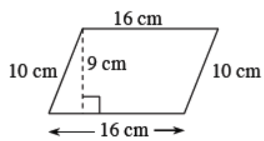
4-42. Alexis, Bart, Chuck, and Dariah all called in to a radio show to get free tickets to a concert. List all the possible orders in which their calls could have been received.

4-43. When she was younger, Mary had to look up at a 68° angle to see into her father's eyes whenever she was standing 15 inches away. How high above the flat ground were her father's eyes if Mary's eyes were 32 inches above the ground?

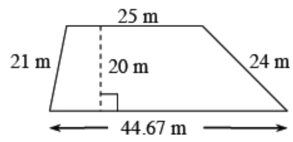
4-44. This problem is a checkpoint for finding areas and perimeters of complex shapes. It will be referred to as Checkpoint 4.

For each figure below, find the area and the perimeter.

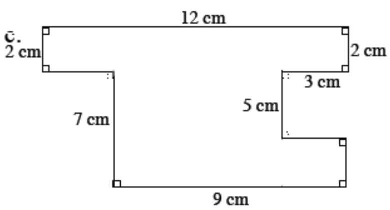
a. Parallelogram



b. Trapezoid



c.



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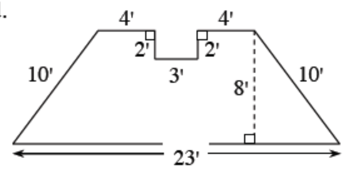
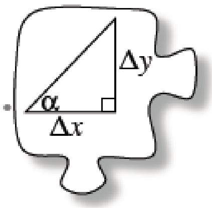


Check your answers by referring to the Checkpoint 4 materials located at the back of your book.

If you needed help solving these problems correctly, then you need more practice. Review the Checkpoint 4 materials and try the practice problems. Also, consider getting help outside of class time. From this point on, you will be expected to do problems like these quickly and easily.

4.1.5 What if I can't measure it?



Applying the Tangent Ratio

In this section so far, you have learned how to find the legs of a right triangle using an angle. But how can you use this information? Today you and your team will use the tangent ratio to solve problems and answer questions.

4-45. STATUE OF LIBERTY

Lindy gets nosebleeds whenever she is more than 300 feet above the ground. During a class fieldtrip, her teacher asked if she wanted to climb to the top of the Statue of Liberty. Since she does not want to get a nosebleed, she decided to take some measurements to figure out the height of the torch of the statue. She found a spot directly under the torch and then measured 42 feet away and determined that the angle up to the torch was 82°. Her eyes are 5 feet above the ground.

Should she climb to the top or will she get a nosebleed? Draw a diagram that fits this situation. Justify your conclusion.

4-46. HOW TALL IS IT?

How tall is Mount Everest? How tall is the White House? Often you want to know a measurement of something you cannot easily measure with a ruler or tape measure. Today you will work with your team to measure the height of something inside your classroom or on your school's campus in order to apply your new tangent tool.

Your Task: Get a **clinometer** (a tool that measures a slope angle) and a meter stick (or tape measure) from your teacher. As a team, decide how you will use these tools to find the height of the object selected by your teacher. Be sure to record all measurements carefully on your Lesson 4.1.5A Resource Page and include a diagram of the situation.

Discussion Points

What should the diagram look like?

What measurements would be useful?

How can you use your tools effectively to get accurate measurements?

Methods and Meanings

Math Notes

Independent Events

Two events are **independent** if knowing that one event occurred does not affect the probability of the other event occurring. For example, one probabilistic situation might be about a vocabulary quiz in science class today with possible outcomes {have a quiz, do not have a quiz}. Another probabilistic situation might be the outcome of this weekend's football game with the possibilities {win, lose, tie}. If you know that a quiz occurred today, it does not change the probability of the football team winning this weekend. The two events are independent.

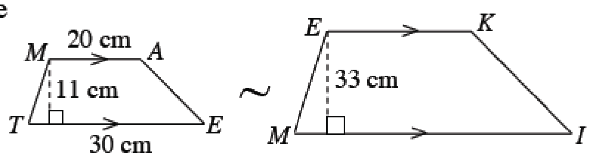
A box contains three red chips and three black chips. If you get a red chip on the first try (and put it back in the box), the probability of getting a red chip on the second try is $\frac{3}{6}$. If you did not get a red chip on the first try, the probability of getting a red chip on the second try is still $\frac{3}{6}$. The probability of getting a red chip on the second try was not changed by knowing whether you got a red chip on the first try or not. When you return the chips to the box, the events {red on first try} and {red on second try} are independent.

However, if an event that occurred changes the probability of another event, the two events are **not independent**. Since getting up late this morning changes the probability that you will eat breakfast, these two events would not be independent.

If you get a red chip on the first try, and *do not put the first chip back in the box*, the probability of a red chip on the second try is $\frac{2}{5}$. If you did not get a red chip on the first try, the probability of getting a red chip on the second try is $\frac{3}{5}$. The probability of getting a red chip on the second try was changed by whether you got a red chip on the first try or not. When you do not replace the chips between draws, the events {red on first try} and {red on second try} are not independent.

Review & Preview

4-47. The trapezoids at right are similar.

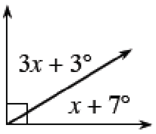


a. What is the ratio of the heights?

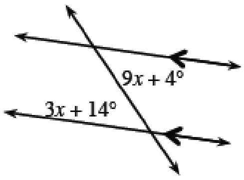
b. Compare the areas. What is the ratio of the areas?

4-48. For each diagram below, write an equation and solve for x, if possible.

a.



b.



4-49. Which of the following events are independent? Refer to the Math Notes in this lesson.

a. Flipping a head, after flipping 5 heads in a row.

b. Drawing an Ace from a deck of playing cards, after two Aces were just drawn (and not returned to the deck).

c. Having blue eyes, if you have blonde hair.

d. The probability of rain this weekend, if the debate team from North City High School wins the state championship.

e. Randomly selecting a diet soda from a cooler filled with both diet and regular soda, after the person before you just selected a diet soda and drank it.

4-50. Leon is standing 60 feet from a telephone pole. As he looks up, a red-tailed hawk lands on the top of the pole. Leon's angle of sight up to the bird is 22° and his eyes are 5.2 feet above the ground.

a. Draw a detailed picture of this situation. Label it with all of the given information.

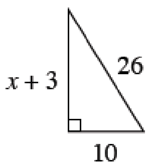
b. How tall is the pole? Show all of your work.

4-51. Examine each sequence below. State whether it is arithmetic, geometric, or neither. For the sequences that are arithmetic or geometric, find the equation for $t(n)$ or a_n.

a. $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$

b. $-7.5, -9.5, -11.5, \ldots$

4-52. Find the value of x in the triangle at right. Refer to problem 4-8 for help. Show all work.



Lesson 4.1.5A Resource Page Team Members: ____________________________

Clinometer Activity

For each object you are measuring, draw a diagram of the object. In the diagram, indicate which measurements your team made and then show all appropriate calculations.

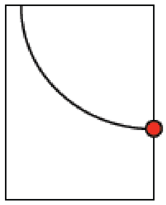
Diagram	Measurements	Calculations
Object #1:		
Object #2:		
Object #3:		

Lesson 4.1.5B Resource Page

How To Make A Clinometer

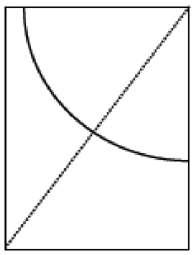
Materials: paper copy of Lesson 4.1.6B Resource Page, straw, string, staplers, 4" by 6" index card (or cardboard for a more durable product), glue, tape, ¼-inch washer.

- Cut out and tape or glue a protractor scale to the index card or cardboard as shown below. **Note** that the zero on the protractor should be on the ***edge*** of the card.



Note: Zero is here.

- Tape a straw along the top of the card, as shown. Put the string into the straw at X, pull it through the straw, and staple or tape it to the card near point Y (see below).

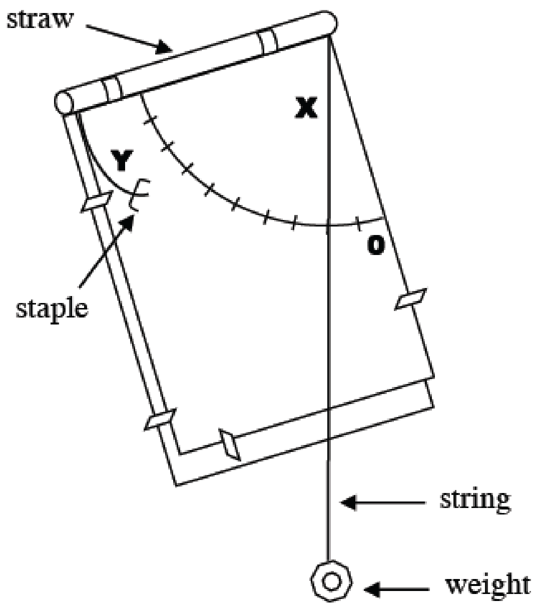


Note: String needs to be at least as long as the diagonal length of the index card.

- Tie a weight (such as a washer) to the other end of the string so that the string will hang vertically a little below the bottom of the card. Ensure that the string is long enough for the weight to hang freely throughout the rotation of the clinometer.

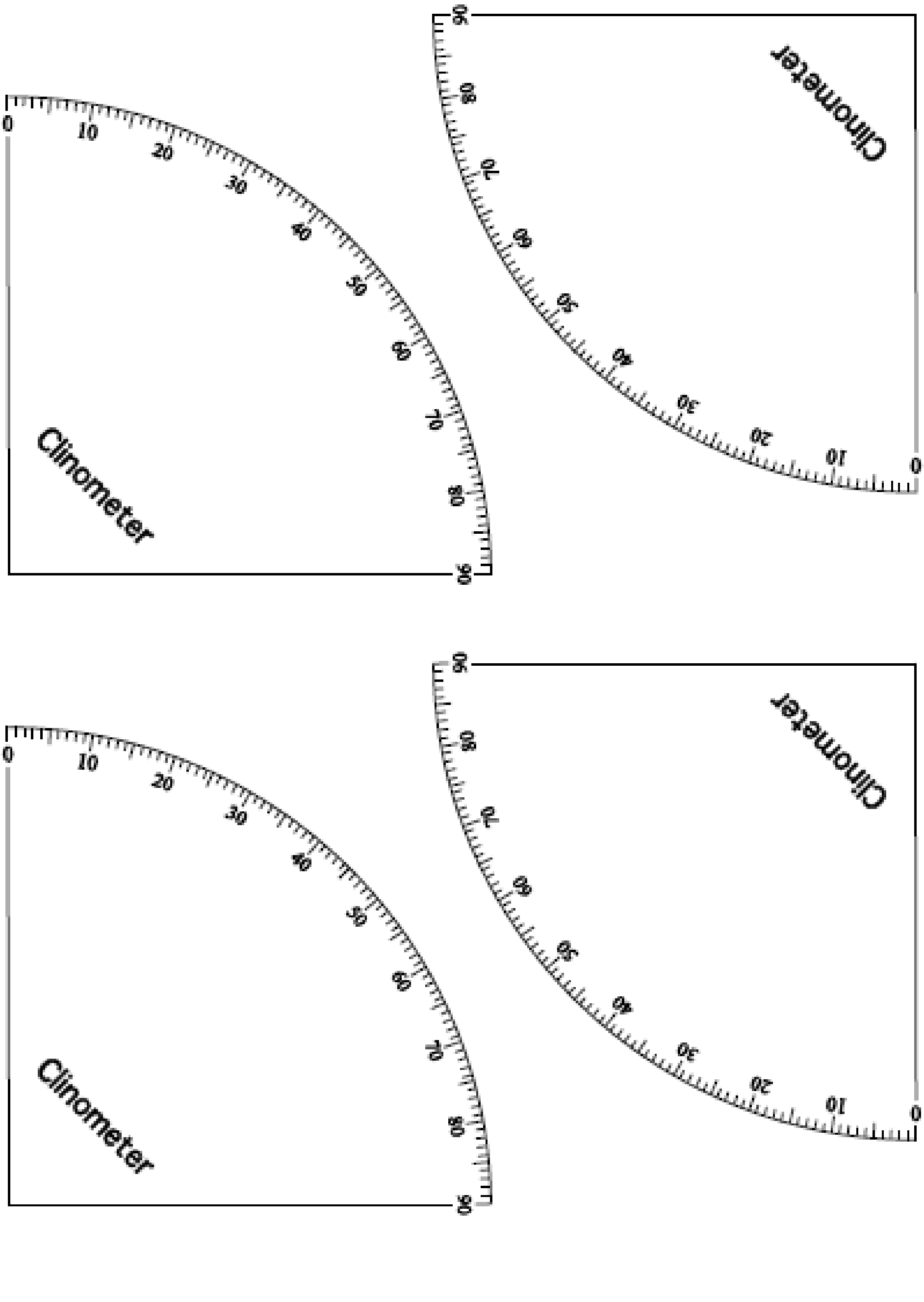
- When using the clinometer, you should stand at a convenient distance from the object that you are measuring. Sight the top of the object through the straw and record the angle where the string intersects the protractor scale. Your partner can help you with this.

* The Clinometer activity is based on a presentation given by Micheal Palmer at the 1987 Invitational Summer Institute of the Northern California Mathematics Project.



Lesson 4.1.5C Resource Page

Clinometer Protractors



4.2.1 How can I represent it?

Using an Area Model

In previous courses you studied probability, which is a measure of the chance that a particular event will occur. In the next few lessons you will encounter a variety of situations that require probability calculations. You will develop new probability tools to help you analyze these situations. The next two lessons focus on tools for listing *all* the possible outcomes of a probability situation, called a **sample space**.

In homework, you have practiced determining probabilities in situations where each outcome you listed had an equal probability of occurring. But what if a game is biased so that some outcomes are more likely than others? How can you represent biased games? Today you will learn a new tool to analyze more complicated situations of chance, called an area model.

4-53. IT'S IN THE GENES


Example of a thumb that can bend backwards at the joint.

Can you bend your thumb backwards at the middle joint to make an angle, like the example at right? Or does your thumb remain straight? The ability to bend your thumb back is thought to rely on a single gene.

What about your tongue? If you can roll your tongue into a "U" shape, you probably have a special gene that enables you to do this.

Assume that half of the U.S. population can bend their thumbs backwards and that half can roll their tongues. Also assume that these genes are independent (in other words, having one gene does not affect whether or not you have the other) and randomly distributed (spread out) throughout the population. Then the sample space of these genetic traits can be organized in a table like the one below.

		Bend thumb?	
		Yes $\frac{1}{2}$	No $\frac{1}{2}$
Roll Tongue?	Yes $\frac{1}{2}$		
	No $\frac{1}{2}$		

a. According to this table, what is the probability that a random person from the U.S. has both special traits? That is, what is the chance that he or she can roll his or her tongue *and* bend his or her thumb back?

b. According to this table, what is the probability that a random person has only one of these special traits? Justify your conclusion.

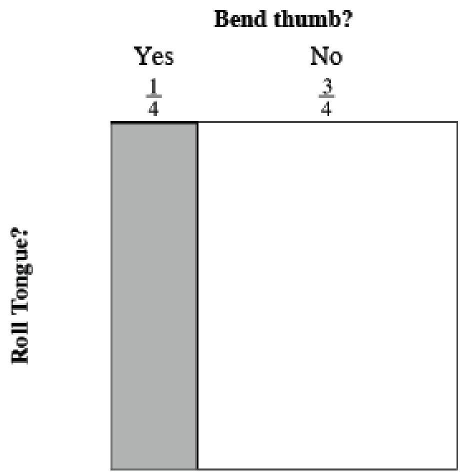
c. This table is useful because every cell in the table is equally likely. Therefore, each possible outcome, such as being able to bend your thumb but not roll your tongue, has a $\frac{1}{4}$ probability.

However, this table assumes that half the population can bend their thumbs backwards, but in reality only about $\frac{1}{4}$ of the U.S. population can bend their thumbs backwards and $\frac{3}{4}$ cannot. It also turns out that a lot more (about $\frac{7}{10}$) of the population can roll their tongues. How can this table be adjusted to represent these percentages? Discuss this with your team and be prepared to share your ideas with the class.

4-54. USING AN AREA MODEL

One way to represent a sample space that has outcomes that are not equally likely is by using a **probability area model**. An area model uses a large square with an area of 1. The square is subdivided into smaller pieces to represent all possible outcomes in the sample space. The area of each outcome is the probability that the outcome will occur.

For example, if $\frac{1}{4}$ of the U.S. population can bend their thumbs back, then the column representing this ability should take only one-fourth of the square's width, as shown at right.



a. How should the diagram be altered to that show that $\frac{7}{10}$ of the U.S. can roll their tongues? Copy this diagram on your paper and add two rows to represent this probability.

b. The relative probabilities for different outcomes are represented by the areas of the regions. For example, the portion of the probability area model representing people with both special traits is a rectangle with a width of $\frac{1}{4}$ and a height of $\frac{7}{10}$. What is the area of this rectangle? This area tells you the probability that a random person in the U.S. has both traits.

c. What is the probability that a randomly selected person can roll his or her tongue but not bend his or her thumb back? Show how you got this probability.

4-55. PROBABILITIES IN VEIN

You and your best friend may not only look different, you may also have different types of blood! For instance, members of the American Navajo population can be classified into two groups: 73% percent (73 out of 100) of the Navajo population has type “O” blood, while 27% (27 out of 100) has type “A” blood. (Blood types describe certain chemicals, called “antigens,” that are found in a person’s blood.)

		Navajo Person #1	
		O $\frac{73}{100}$	**A** $\frac{27}{100}$
Navajo Person #2	**O** $\frac{73}{100}$		
	A $\frac{27}{100}$		

a. Suppose you select two Navajo individuals at random. What is the probability that both individuals have type “A” blood? This time, drawing an area model that is exactly to scale would be challenging. A probability area model (like the one above) is still useful because it will still allow you to calculate the individual areas, even without drawing it to scale. Copy and complete this “generic” probability area model.

b. What is the probability that two Navajo individuals selected at random have the same blood type?

4-56. SHIPWRECKED!

Zack and Nick (both from the U.S.) are shipwrecked on a desert island! Zack has been injured and is losing blood rapidly, and Nick is the only person around to give him a transfusion.

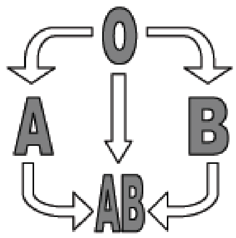
Unlike the Navajo you learned of in problem 4-55, most populations are classified into four blood types: O, A, B, and AB. For example, in the U.S., 45% of people have type O blood, 40% have type A, 11% have type B, and 4% have type AB (according to the American Red Cross, 2004). While there are other ways in which people's blood can differ, this problem will only take into account these four blood types.

a. Make a probability area model representing the blood types in this problem. List Nick's possible blood types along the top of the model and Zack's possible blood types along the side.

	O (45%)	A (40%)	B (11%)	AB (4%)
O (45%)				
A (40%)				
B (11%)				
AB (4%)				

b. What is the probability that Zack and Nick have the same blood type?

c. Luckily, two people do not have to have the same blood type for the receiver of blood to survive a transfusion. Other combinations will also work, as shown in the diagram at right. Assuming that their blood is compatible in other ways, a donor with type O blood can donate to receivers with type O, A, B, or AB, while a donor with type A blood can donate to a receiver with A or AB. A donor with type B blood can donate to a receiver with B or AB, and a donor with type AB blood can donate only to AB receivers.



Assuming that Nick's blood is compatible with Zack's in other ways, determine the probability that he has a type of blood that can save Zack's life!

4-57. You made a critical assumption in problem 4-56 when you made a probability area model and multiplied the probabilities.

a. Blood type is affected by genetic inheritance. What if Zack and Nick were related to each other? What if they were brothers or father and son? How could that affect the possible outcomes?

b. What has to be true in order to assume a probability area model will give an accurate theoretical probability?

METHODS AND MEANINGS

MATH NOTES

Solving a Quadratic Equation

In a previous course, you learned how to solve **quadratic equations** (equations that can be written in the form $ax^2 + bx + c = 0$). Review two methods for solving quadratic equations below.

Some quadratic equations can be solved by **factoring** and using the **Zero Product Property**. For example, because $x^2 - 3x - 10 = (x-5)(x+2)$, the quadratic equation $x^2 - 3x - 10 = 0$ can be rewritten as $(x-5)(x+2) = 0$. The Zero Product Property states that if $ab = 0$, then $a = 0$ or $b = 0$. So, if $(x-5)(x+2) = 0$, then $x - 5 = 0$ or $x + 2 = 0$. Therefore, $x = 5$ or $x = -2$.

Another method for solving quadratic equations is the **Quadratic Formula**. This method is particularly helpful for solving quadratic equations that are difficult or impossible to factor. Before using the Quadratic Formula, the quadratic equation you want to solve must be in standard form, that is, written as $ax^2 + bx + c = 0$.

In this form, a is the coefficient of the x^2 term, b is the coefficient of the x term, and c is the constant term. The Quadratic Formula states:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

This formula gives two possible answers due to the "$\pm$" symbol. This symbol (read as "plus or minus") is shorthand notation that tells us to calculate the formula twice: once using addition and once using subtraction. Therefore, every Quadratic Formula problem must be simplified twice to give:

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{or} \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

To solve $x^2 - 3x - 10 = 0$ using the Quadratic Formula, substitute $a = 1$, $b = -3$, and $c = -10$ into the formula, as shown below.

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(-10)}}{2(1)} \Rightarrow \frac{3 \pm \sqrt{49}}{2} \Rightarrow \frac{3+7}{2} \text{ or } \frac{3-7}{2} \Rightarrow x = 5 \text{ or } x = -2$$

Review & Preview

4-58. Out of the 20 contestants in the state math championships, 10 are girls. For this round, each contestant gets asked one question. The first question goes to a randomly chosen contestant.

a. What is the probability the first contestant is a girl?

b. If the first contestant is a girl, what is the probability that the second contestant is a girl?

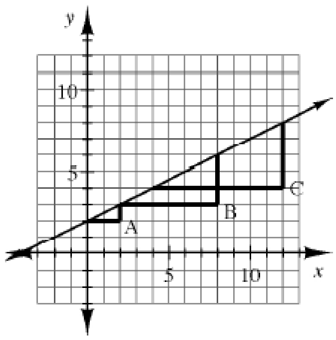
c. Is the probability that the second contestant a girl independent of the first contestant being a girl? Refer to the Math Notes box at the end of Lesson 4.1.5.

4-59. On graph paper, graph the parabola $y = 2x^2 - 5x - 3$.

a. What are the roots (x-intercepts) of the parabola?

b. Read the Math Notes box for this lesson. Then solve the equation $2x^2 - 5x - 3 = 0$ algebraically. Did you solutions match your roots from part (a)?

4-60. Examine the graph at right with slope triangles A, B, and C.



a. Find the slope of the line using slope triangle A, slope triangle B, and then slope triangle C.

b. Hernisha's slope triangle has a slope of $\frac{1}{2}$. What do you know about her line?

4-61. Francis and John are racing. Francis is 2 meters in front of the starting line at time $t = 0$ and he runs at a constant rate of 1 meters per second. John is 5 meters in front of the starting line and he runs at a constant rate of 0.75 meters per second. After how long will Francis catch up to John?

4-62. Solve each equation to find the value of x. Leave your answers in decimal form accurate to the thousandths place.

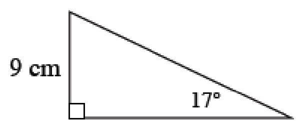
a. $\frac{3.2}{x} = \frac{7.5}{x^2}$

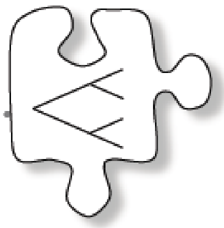
b. $4(x-2) + 3(-x+4) = -2(x-3)$

c. $2x^2 + 7x - 15 = 0$

d. $3x^2 - 2x = -1$

4-63. Find the perimeter of the shape at right. Clearly show all your steps.





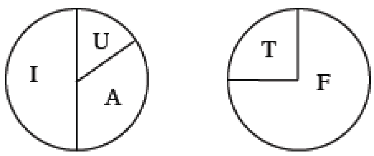
4.2.2 How can I represent it?

Using a Tree Diagram

In Lesson 4.2.1, you used a probability area model to represent probability situations where some outcomes were more likely than others. Today you will consider how to represent these types of situations using tree diagrams.

4-64. Your teacher challenges you to a spinner game. You spin the two spinners with the probabilities listed at right. The first letter comes from Spinner #1 and the second letter from Spinner #2. If the letters can form a two-letter English word, you win. Otherwise, your teacher wins.

Spinner #1 **Spinner #2**



$P(I)=\frac{1}{2}$ $P(T)=\frac{1}{4}$

$P(U)=\frac{1}{6}$ $P(F)=\frac{3}{4}$

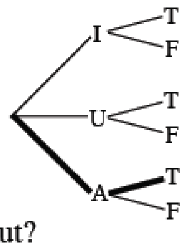
$P(A)=\frac{1}{3}$

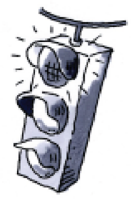
a. Are the outcomes for spinner #2 independent of the outcomes on spinner #1?

b. Make a probability area model of the sample space, and find the probability that you will win this game.

c. Is this game fair? If you played the game 100 times, who do you think would win more often, you or your teacher? Can you be sure this will happen?

4-65. Sinclair wonders how to model the spinner game in problem 4-64 using a tree diagram. He draws the tree diagram at right.





a. Sabrina says, "*That can't be right. This diagram makes it look like all the words are equally likely.*" What is Sabrina talking about? Why is this tree diagram misleading?

b. To make the tree diagram reflect the true probabilities in this game, Sabrina writes numbers on each branch showing the probability that the letter will occur. So she writes a "$\frac{1}{3}$" on the branch for "A," a "$\frac{1}{4}$" on the branch for each "T," etc. Following Sabrina's method, label the tree diagram with probabilities on each branch.

c. According to the probability area model that you made in problem 4-64, what is the probability that you will spin the word "AT"? Now examine the bolded branch on the tree diagram shown above. How could the numbers you have written on the tree diagram be used to find the probability of spinning "AT"?

d. Does this method work for the other combinations of letters? Similarly calculate the probabilities for each of the paths of the tree diagram. At the end of each branch, write its probability. (For example, write "$\frac{1}{12}$" at the end of the "AT" branch.) Do your answers match those from problem 4-64?

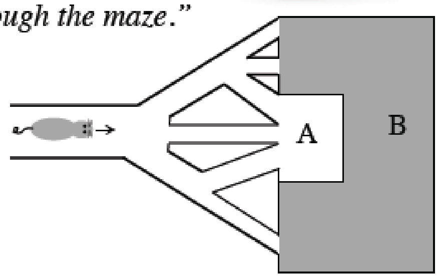
e. Find all the branches with letter combinations that make words. Use the numbers written at the end of each branch to compute the total probability that you will spin a word. Does this probability match the probability you found with your area model?

4-66. THE RAT RACE

Ryan has a pet rat Romeo that he boasts is the smartest rat in the county. Sammy overheard Ryan at the county fair claiming that Romeo could learn to run a particular maze and find the cheese at the end.

"*I don't think Romeo is that smart!*" Sammy declares, "*I think the rat just chooses a random path through the maze*."

Ryan has built a maze with the floor plan shown at right. In addition, he has placed some cheese in an airtight container (so Romeo can't smell the cheese!) in room A.



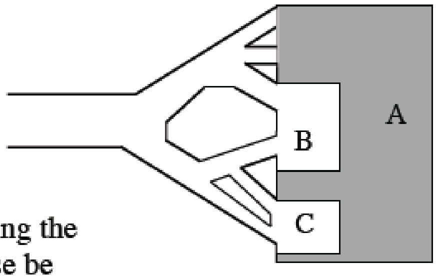
a. Suppose that every time Romeo reaches a split in the maze, he is equally likely to choose any of the paths in front of him. Choose a method and calculate the probability that Romeo will end up in each room. In a sentence or two, explain why you chose the method you did.

b. If the rat moves through the maze randomly, how many out of 100 attempts would you expect Romeo to end up in room A? How many times would you expect him to end up in room B? Explain.

c. After 100 attempts, and Romeo finds the cheese 66 times. "*See how smart Romeo is?*" Ryan asks, "*He clearly learned something and got better at the maze as he went along*." Sammy isn't so sure.

Do you think Romeo learned and improved his ability to return to the same room over time? Or could he just have been moving randomly? Discuss this question with your team. Then, write an argument that would convince Ryan or Sammy.

4-67. Always skeptical, Sammy says, "*If Romeo really can learn, he ought to be able to figure out how to run this new maze I've designed.*" Examine Sammy's maze at right.



a. To give Romeo the best chance of finding the cheese, in which room should the cheese be placed? Choose a method, show all steps in your solution process, and justify your answer.

b. If the cheese is in room C and Romeo finds the cheese 6 times out of every 10 tries, does he seem to be learning? Explain your conclusion.

4-68. Make an entry in your Learning Log describing the various ways of representing complete sample spaces. For each method, indicate how you compute probabilities using the method. Which method seems easiest to use so far? Label this entry "Creating Sample Spaces" and include today's date. Set this Learning Log aside in a safe place. You will need it in the next lesson.

Review & Preview

4-69. Eddie is arguing with Tana about the probability of flipping three coins. They decided to flip a penny, nickel, and a dime.

a. Which would be better for determining the sample space, a tree diagram or an area model? Justify your answer.

b. Make a sample space that shows all the possible outcomes. How many outcomes are there?

c. Find the probability of each of the following events occurring. Be sure to show your thinking clearly:

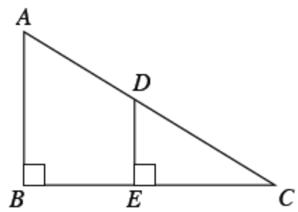
i. Three heads   *ii.* One head and two tails

iii. At least one tail   *iv.* Exactly two tails

d. Which is more likely, flipping at least 2 heads or at least 2 tails? Explain.

e. How would the probabilities change if Tana found out that Eddie was using weighted coins (coins that were not fair) so that the probability of getting heads for each coin was $\frac{4}{5}$ instead of $\frac{1}{2}$? Would this change the sample space? Recalculate the probabilities in part (c) based on the new information.

4-70. Are the triangles at right similar? If so, write a flowchart that justifies your conclusion. If not, explain how you know.



4-71. You roll a die and it comes up a "6" three times in a row. What is the probability of rolling a "6" on the next toss?

4-72. Mr. Singer made the flowchart at right about a student named Brian.

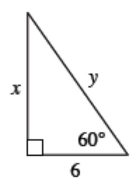
a. What is wrong with Mr. Singer's flowchart?

b. Rearrange the ovals so the flowchart makes more sense.

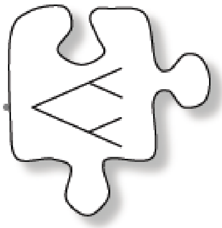
4-73. Write the first four terms of each of the following sequences.

a. $a_n = 3 \cdot 5^{n-1}$ b. $a_1 = 10,\ a_{n+1} = -5a_n$

4-74. Find x and y in the diagram at right. Show all of the steps leading to your answer.



4.2.3 What model should I use?



Probability Models

In this lesson you will review ideas of probability as you use systematic lists, tree diagrams, and area models to account for all of the elements in a sample space, account for equally likely outcomes, and identify events. You will find that certain tools may work better for particular situations. In one problem a tree diagram or list might be most efficient, while in another problem an area model may be the best choice. As you work with your team, keep the following questions in mind:

What are the possible outcomes?

Are the outcomes equally likely?

Will a tree diagram, list, or area model help?

What is the probability for this event?

4-75. ROCK, PAPER, SCISSORS

Your team will play a variation of "Rock, Paper, Scissors" (sometimes called "Rochambeau") and record points. You will need to work in a team of four. Have one person act as recorder while the other three play the game.

a. List the names of the people in your team alphabetically. The first person on the list is Player A, the next is Player B, the third is Player C, and the fourth is the recorder. Write down who has each role.

b. Without playing the game, discuss with your team which player you think will receive the most points by the end of the game. Assign points as follows:

- Player A gets a point each time all three players match.
- Player B gets a point each time two of the three players match.
- Player C gets a point each time none of the players match.

c. Now play "Rock, Paper, Scissors" with your team at least 20 times. The recorder should record the winner for each round. Does this game seem fair?

d. Calculate the theoretical probability for each outcome (Player A, Player B, or Player C winning). Discuss this with your team and be prepared to share your results with the class.

e. Devise a plan to make this game fair.

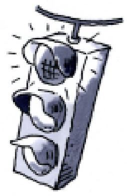
4-76. There is a new game at the school fair called "Pick a Tile," in which the player reaches into two bags and chooses one square tile and one circular tile. The bag with squares contains three yellow, one blue, and two red squares. The bag with circles has one yellow and two red circles. In order to win the game (and a large stuffed animal), a player must choose one blue square and one red circle.

Since it costs $2 to play the game, Marty and Gerri decided to calculate the probability of winning before deciding whether to play.

Gerri suggested making a systematic list of all the possible color combinations in the sample space, listing squares first then circles:

RY *BY* *YY*
RR *BR* *YR*

"So," says Gerri, *"the answer is* $\frac{1}{6}$*."*



"That doesn't seem quite right," says Marty. *"There are more yellow squares than blue ones. I don't think the chance of getting a yellow square and a red circle should be the same as getting a blue square and a red circle."*

a. Make a tree diagram for this situation. Remember to take into account the duplicate tiles in the bags.

b. Find the probability of a player choosing the winning blue square-red circle combination.

c. Should Gerri and Marty play this game? Would you? Why or why not?

4-77. Now draw a probability area model for the "Pick a Tile" game in problem 4-76.

a. Use the probability area model to calculate the probability of each possible color combination of a square and a circular tile.

b. Explain to Marty and Gerri why the probability area model is called an *area* model.

c. Discuss which model you preferred using to solve the "Pick a Tile" problem with your team. What are your reasons for your preference?

d. Could you have used the area model for the "Rock, Paper, Scissors" problem? Explain why or why not.

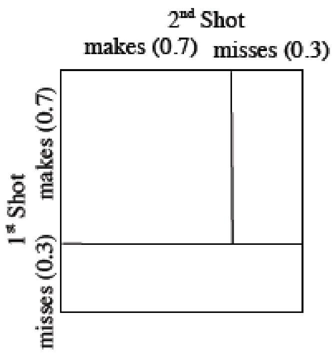
4-78. BASKETBALL: Shooting One-and-One Free Throws

Rimshot McGee has a 70% free throw average. The opposing team is ahead by one point. Rimshot is at the foul line in a one-and-one situation with just seconds left in the game. (A one-and-one situation means that the player shoots a free throw. If they make the shot, they are allowed to shoot another. If they miss the first shot, they get no second shot. Each shot made is worth one point.)

a. First, take a guess. What do you think is the most likely outcome for Rimshot: zero points, one point, or two points?

b. Draw a tree diagram to represent this situation.

c. Jeremy is working on the problem with Jenna and he remembers that area models are sometimes useful for solving problems related to probability. They set up the probability area model at right. Discuss this model with your team. Which part of the model represents Rimshot getting one point? How can you use the model to help calculate the probability that Rimshot will get exactly one point?



d. Use either your tree diagram or the area model to help you calculate the probabilities that Rimshot will get either 0 or 2 points. What is the most likely of the three outcomes?

4-79. With your team, examine the probability area model from problem 4-78.

a. What are the dimensions of the large rectangle? Explain why these dimensions make sense.

b. What is the total area of the model? Express the area as a product of the dimensions and as a sum of the parts.

c. What events are represented by the entire area model?

4-80. This Learning Log extends the entry that you made in problem 4-68. In that entry you described the various ways of representing complete sample spaces and showed how to use each method to find probabilities.

Expand upon your entry. Are there any conditions under which certain methods to represent the sample space can or cannot be used? Which methods seem most versatile? Why? Title this entry "Conditions For Using Probability Methods" and include today's date.

Methods and Meanings

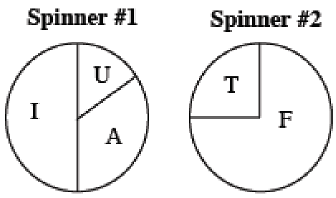
MATH NOTES

Probability Models

When all the possible outcomes of a probabilistic event are *equally likely*, you can calculate probabilities as follows:

$$\text{Theoretical probability} = \frac{\text{number of successful outcomes}}{\text{total number of possible outcomes}}$$

But suppose you spin the two spinners shown to the right. These outcomes are not all equally likely so another model is needed to calculate probabilities of outcomes.

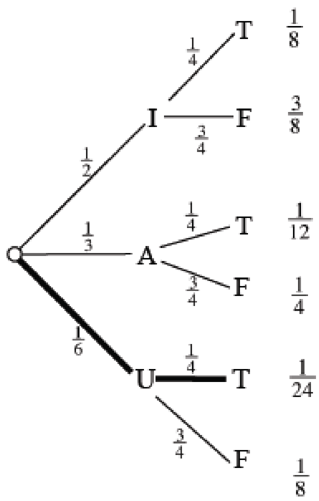


Spinner #1

Spinner #2	I $\left(\frac{1}{2}\right)$	A $\left(\frac{1}{3}\right)$	U $\left(\frac{1}{6}\right)$
T $\left(\frac{1}{4}\right)$	IT $\left(\frac{1}{8}\right)$	AT $\left(\frac{1}{12}\right)$	UT $\left(\frac{1}{24}\right)$
F $\left(\frac{3}{4}\right)$	IF $\left(\frac{3}{8}\right)$	AF $\left(\frac{1}{4}\right)$	UF $\left(\frac{1}{8}\right)$

A **probability area model** is practical if there are exactly two probabilistic situations and they are independent. The outcomes of one probabilistic situation are across the top of the table, and the outcomes of the other are on the left. The smaller rectangles are the sample space. Then the probability for an outcome is the area of the rectangle. For example, the probability of spinning "UT" is $\frac{1}{6} \cdot \frac{1}{4} = \frac{1}{24}$. Notice that the area (the probability) of the large overall square is 1.

A **tree diagram** can be used even if there are more than two probabilistic situations, and the events can be independent or not. In this model, the ends of the branches indicate outcomes of probabilistic situations, and the branches show the probability of each event. For example, in the tree diagram at right the first branching point represents Spinner #1 with outcomes "I" "A" or "U". The numbers on the branch represent the probability that a letter occurs.



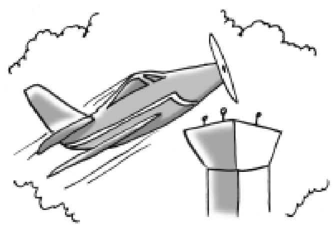
The numbers at the far right of the table represent the probabilities of various outcomes. For example, the probability of spinning "U" and "T" can be found at the end of the bold branch of the tree. This probability, $\frac{1}{24}$, can be found by multiplying the fractions that appear on the bold branches.

Review & Preview

4-81. Eddie told Alfred, *"I'll bet if I flip three coins I can get exactly two heads."* Alfred replied, *"I'll bet I can get exactly two heads if I flip four coins!"* Eddie scoffed, *"Well, so what? That's easier."* Alfred argued, *"No, it's not. It's harder."* Who is correct? Show all of your work and be prepared to defend your conclusion.

4-82. Find the equation of the line with a slope of $\frac{1}{3}$ that goes through the point $(0, 9)$.

4-83. An airplane takes off and climbs at an angle of $11°$. If the plane must fly over a 120-foot tower with at least 50 feet of clearance, what is the minimum distance between the point where the plane leaves the ground and the base of the tower?



a. Draw and label a diagram for this situation.

b. What is the minimum distance between the point where the plane leaves the ground and the tower? Explain completely.

4-84. Solve each equation below for the given variable. Show all work and check your answer.

a. $\sqrt{x} - 5 = 2$

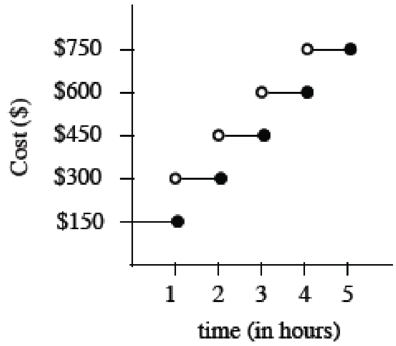
b. $-4(-2 - x) = 5x + 6$

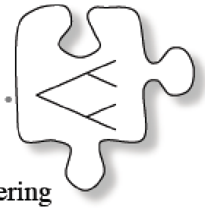
c. $\frac{5}{x-2} = \frac{3}{2}$

d. $x^2 + 4x - 5 = 0$

4-85. Can a triangle be made with sides of length 7, 10, and 20 units? Justify your answer.

4-86. According to the graph at right, how much money would it cost to speak to an attorney for 2 hours and 25 minutes?





4.2.4 What if both events happen?

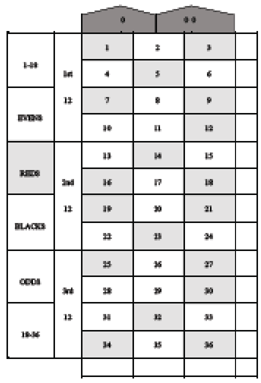
Unions, Intersections, and Complements

In the mid 1600's, a French nobleman, the Chevalier de Mere, was wondering why he was losing money on a bet that he thought was a sure winner. He asked the mathematician Blaise Pascal, who consulted with another mathematician, Pierre de Fermat. Together they solved the problem, and this work provided a beginning for the development of probability theory. Since argument over the analysis of a dice game provided a basis for the study of this important area of mathematics, casino games are a reasonable place to continue to investigate and clarify the ideas and language of probability.

As one of the simplest casino games to analyze, roulette is a good place to start. In American roulette the bettor places a bet, the croupier (game manager) spins the wheel and drops the ball and then everyone waits for the ball to land in one of 38 slots. The 38 slots on the wheel are numbered 00, 0, 1, 2, 3, …, 36. Eighteen of the numbers are red and eighteen are black; 0 and 00 are green. (In French roulette, also known as Monte Carlo, there is no 00, so there are only 37 slots on the wheel.)

Before the ball is dropped, players place their chips on the roulette layout, shown at right. Bets can be placed on:

- A single number;
- Two numbers by placing the chip on the line between them;
- Three numbers by placing a chip on the line at the edge of a row of three;
- Four numbers by placing the chip where the four corners meet;
- Five numbers (0, 00, 1, 2, 3);
- Six numbers by placing the bet at an intersection at the edge;
- A column, the 1st twelve, 2nd twelve, or 3rd twelve;
- Even numbers;
- Odds numbers;
- 1-18;
- 19-36;
- Red numbers; or,
- Black numbers



Note: The lightly shaded numbers, 1, 3, 5, 7, 9, 12, 14, 16, 18, 19, 21, 23, 25, 27, 30, 32, 34, and 36 are red.

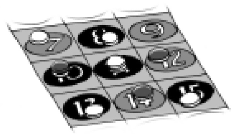
4-87. Obtain a Lesson 4.2.4A Resource Page from your teacher. On the resource page, the "chips" A through K represent possible bets that could be made.

a. What is the sample space for one spin of the roulette wheel?

b. Are the outcomes equally likely?

c. A subset (smaller set) of outcomes from the sample space is called an event. For example, chip A represents the event {30}, and chip B represents the event {22, 23}. Make a list of events and their probabilities for Chips A-K.

4-88. Some roulette players like to make two (or more) bets at the same time. A bettor places a chip on the event {7, 8, 10, 11} and then another chip on the event {10, 11, 12, 13, 14, 15}. What numbers will allow the bettor to win both bets? Next find the bettor's chances of winning the bet of the first chip *and* winning the bet of the second chip on a single drop of the roulette ball. This called is the probability of the ball landing on a number that is in the **intersection** of the two events.



4-89. When placing two different bets, most players are just hoping that they will win on one *or* the other of the two events. The player is betting on the **union** of two events.

a. Calculate the probability of winning either one bet on the event {7, 8, 10, 11} *or* another on the event {10, 11, 12, 13, 14, 15}. Think carefully about which set of outcomes that will allow the bettor to win either of the bets when calculating the probability. This probability is the union of the two events.

b. Calculate the probability of the union of {numbers in first column} and {"2^{nd} 12" numbers 13 through 24}.

c. One bettor's chip is on the event {13, 14, 15, 16, 17, 18} and another on {Reds}. What is the probability of the union of these events?

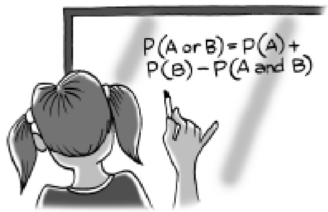
d. Explain your method for finding the probabilities in parts (a) through (c).

4-90. Viola described the following method for finding the probability for part (a) of problem 4-89:

"When I looked at the probability of either of two events, I knew that would include all of the numbers in both events, but sometimes some numbers might be counted twice. So, instead of just counting up all of the outcomes, I added the two probabilities together and then subtracted the probability of the overlapping events or numbers. So it's just $\frac{4}{38}+\frac{6}{38}-\frac{2}{30}=\frac{8}{38}$*."*

Does Viola's method always work? Why or why not? Is this the method you that used to do problem 4-89? If not show how to use Viola's method on one of the other parts of problem 4-89.

4-91. Viola's method of "adding the two probabilities and subtracting the probability of the overlapping event" is called the **Addition Rule** and can be written:



P(A *or* B) = P(A) + P(B) – P(A *and* B)

You have already seen that any event that includes event A *or* event B can be called a union, and is said "A union B." The event where both events A *and* B occur together is called an intersection. So the Addition Rule can also be written:

P(A union B) = P(A) + P(B) – P(A intersection B)

Use these ideas to do the following: A player places a chip on the event {1-18} and another on the event {Reds}. Consider the event {1-18} as event "A", and the event {Reds} as event "B." Clearly show two different ways to figure out the probability of the player winning one of the two bets.

4-92. On the Lesson 4.2.4A Resource Page, consider a player who puts a chip on both events "G" and "I."

a. How does the event {G *or* I} differ from the event {G *and* I}?

b. List the set of outcomes for the intersection of events G and I, and the set of outcomes for the union of events G or I.

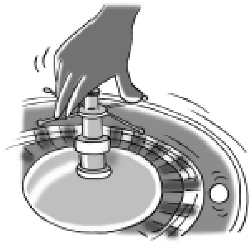
c. Is the player who puts a chip on both G and I betting on the "or" or the "and?" Use both the counting method and the Addition Rule to find the probability that this player will win.

4-93. Wyatt places a bet on event G.

a. What is the probability that he will lose?

b. How did you calculate the probability of {not event G}?

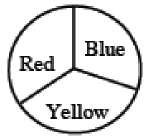
c. Show another method for calculating the probability of the bettor losing on event G.



4-94. Sometimes it is easier to figure out the probability that something will *not* happen than the probability that it *will*. When finding the probability that something will not happen, you are looking at the **complement** of an event. The complement is the set of all outcomes in the sample space that are not included in the event.

Show two ways to solve the problem below, then decide which way you prefer and explain why.

a. Crystal is spinning the spinner at right and claims she has a good chance of having the spinner land on red at least once in three tries. What is the probability that the spinner will land on red at least once in three tries



b. If the probability of an event A is represented symbolically as P(A), how can you symbolically represent the probability of the complement of event A?

METHODS AND MEANINGS

MATH NOTES

Unions, Intersections, and Complements

A smaller set of outcomes from a sample space is called an **event**. For example, if you draw one card from a standard deck of 52 cards, the sample space would be {A♠, A♣, A♥, A♦, 2♠, 2♣, 2♥, 2♦, … , K♠, K♣, K♥, K♦}. An event might be {drawing a spade}, which would be set {A♠, 2♠, 3♠, … , Q♠, K♠}. The event {drawing a face card} is the set {J♠, J♣, J♥, J♦, Q♠, Q♣, Q♥, Q♦, K♠, K♣, K♥, K♦}.

The **complement** of an event is all the outcomes in the sample space that are not in the original event. For example, the complement of {drawing a spade} would be all the hearts, diamonds, and clubs, represented as the complement of {drawing a spade} = {A♥, 2♥, 3♥, …, Q♥, K♥, A♦, 2♦, 3♦, … , Q♦, K♦, A♣, 2♣, 3♣, … , Q♣, K♣}.

The **intersection** of two events is the event in which *both* the first event *and* the second event occur. The intersection of the events {drawing a spade} and {face card} would be {J♠, Q♠, K♠} because these three cards are in both the event {drawing a spade} *and* the event {face card}.

The **union** of two events is the event in which the first event *or* the second event (or both) occur. The union of the events {drawing a spade} and a {face card} is {A♠, 2♠, 3♠, 4♠, 5♠, 6♠, 7♠, 8♠, 9♠, 10♠, J♠, Q♠, K♠, J♣, J♥, J♦, Q♣, Q♥, Q♦, K♣, K♥, K♦}. This event has 22 outcomes.

The probability of *equally likely* events can be found by:

$$P(\text{event}) = \frac{\text{number of successful outcomes}}{\text{total number of possible outcomes}}$$

The probability of {drawing a spade} or {drawing a face card} is $\frac{22}{52}$ because there are 22 cards in the union and 52 cards in the sample space.

Math Notes box continues on next page →

MATH NOTES

METHODS AND MEANINGS

Unions, Intersections, and Complements

Alternatively, the probability of the union of two events can be found by using the **Addition Rule**:

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

or

$$P(A \text{ union } B) = P(A) + P(B) - P(A \text{ intersection } B)$$

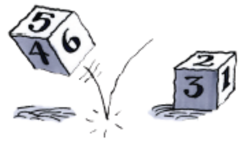
If you let event A be {drawing a spade} and event B be {drawing a face card}, P(A) = P(spade) = $\frac{13}{52}$, P(B) = P(face card) = $\frac{12}{52}$, P(A and B) = P(spade and face card) = $\frac{3}{52}$.

Then, the probability of drawing a spade *or* a face card is:

$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B) = \frac{13}{52} + \frac{12}{52} - \frac{3}{52} = \frac{22}{52}$.

Review & Preview

4-95. Use an area model, a tree diagram, or refer to the table you created in problem 4-10 that represents the sample space for the sum of the numbers when rolling two standard six-sided dice.



a. In a standard casino dice game the roller wins on the first roll if he rolls a sum of 7 or 11. What is the probability of winning on the first roll?

b. The player loses on the first roll if he rolls a sum of 2, 3, or 12. What is the probability of losing on the first roll?

c. If the player rolls any other sum, he continues to roll the dice until the first sum he rolled comes up again or until he rolls a 7, whichever happens sooner. What is the probability that the game continues after the first roll?

4-96. A player in the casino dice game described in problem 4-10 rolled a sum of 6 on his first roll. He will win if he rolls a sum of six on the second roll but lose if he rolls a sum of seven. If anything else happens they ignore the result and he gets to roll again.

a. How many ways are there to get a sum of six?

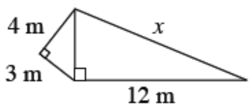
b. How many ways are there to get a sum of seven?

c. How many possible outcomes are important in this problem?

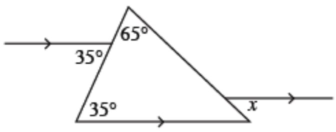
d. What is the probability of getting a sum of six before a sum of seven?

4-97. For each diagram below, solve for x. Name the relationship(s) you used. Show all work.

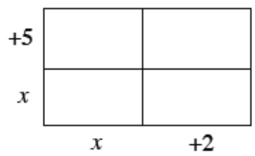
a.



b.

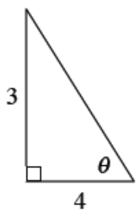


4-98. The area of the rectangle shown at right is 40 square units. Write and solve an equation to find x. Then find the dimensions of the rectangle.

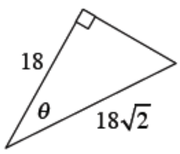


4-99. Based on the measurements provided for each triangle below, decide if the angle θ must be more than, less than, or equal to 45°. Assume the diagram is not drawn to scale. Show how you know.

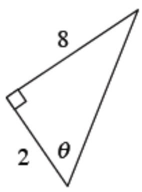
a.



b.

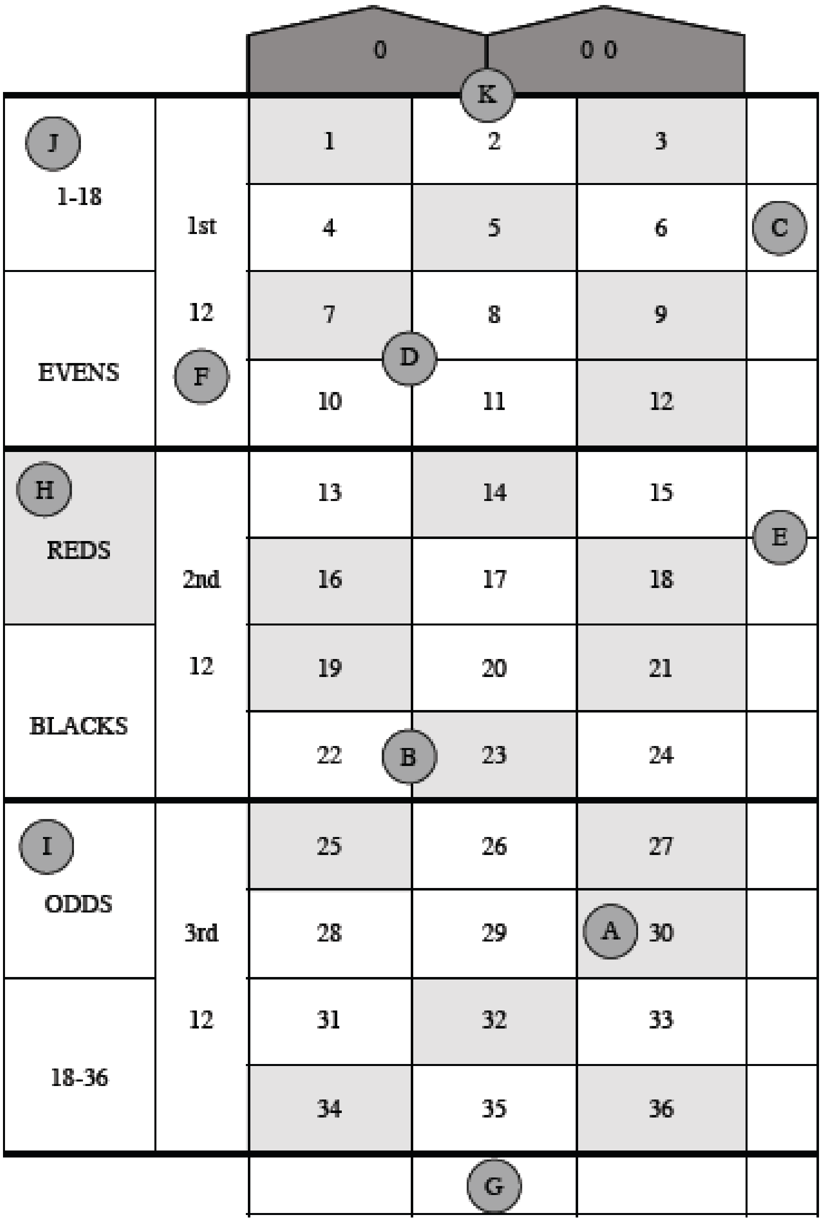


c.

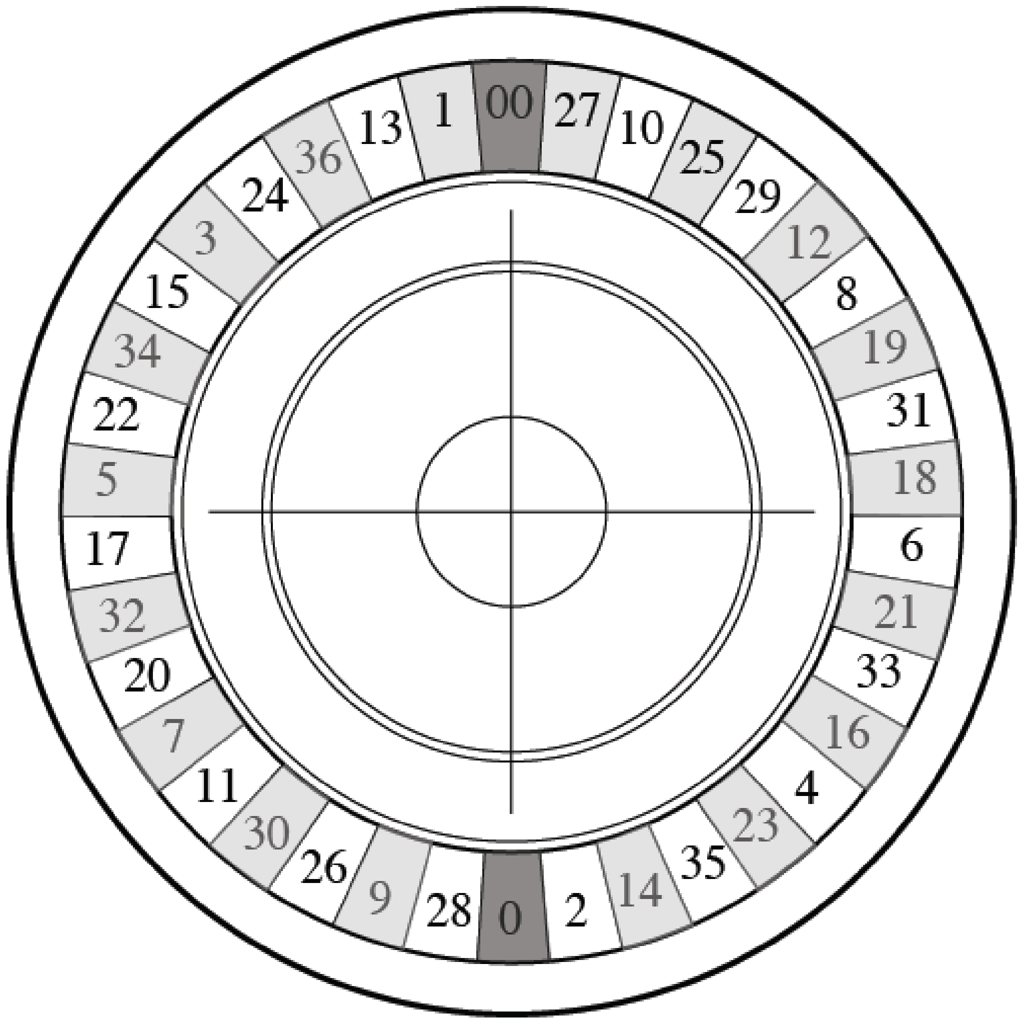


4-100. Find the slope of the line through the points (–5, 86) and (95, 16). Then find at least one more point on the line.

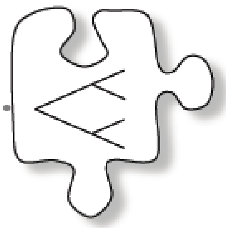
Lesson 4.2.4A Resource Page



Lesson 4.2.4B Resource Page



4.2.5 How much can I expect to win?



Expected Value

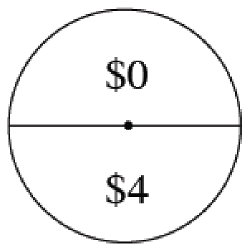
Different cultures have developed creative forms of games of chance. For example, native Hawaiians play a game called Konane, which uses markers and a board and is similar to checkers. Native Americans play a game called To-pe-di, in which tossed sticks determine how many points a player receives.

When designing a game of chance, attention must be given to make sure the game is fair. If the game is not fair, or if there is not a reasonable chance that someone can win, no one will play the game. In addition, if the game has prizes involved, care needs to be taken so that prizes will be distributed based on their availability. In other words, if you only want to give away one grand prize, you want to make sure the game is not set up so that 10 people win the grand prize!

Today your team will analyze different games to learn about expected value, which helps to predict the result of a game of chance.

4-101. TAKE A SPIN

Consider the following game: After you spin the wheel at right, you win the amount spun.

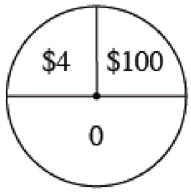


a. If you play the game 10 times, how much money would you expect to win? What if you played the game 30 times? 100 times? Explain your process.

b. What if you played the game n times? Write an equation for how much money someone can expect to win after playing the game n times.

c. If you were to play only once, what would you expect to earn according to your equation in part (b)? Is it actually possible to win that amount? Explain why or why not.

4-102. What if the spinner looks like the one at right instead?

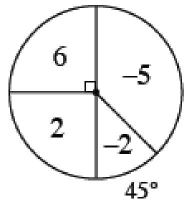


a. If you win the amount that comes up on each spin, how much would you expect to win after 4 spins? What about after 100 spins?

b. Find this spinner's **expected value**. That is, what is the expected amount you will win for each spin? Be ready to justify your answer.

c. Gustavo describes his thinking this way: *"Half the time, I'll earn nothing. One-fourth the time, I'll earn $4 and the other one-fourth of the time I'll earn $100. So, for one spin, I can expect to win* $\frac{1}{2}(0)+\frac{1}{4}(\$4)+\frac{1}{4}(\$100)$ *."* Calculate Gustavo's expression. Does his result match your result from part (b)?

4-103. Jesse has created the spinner at right. This time, if you land on a positive number, you win that amount of money. However, if you land on a negative number, you lose that amount of money! Want to try it?



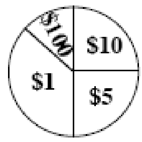
a. Before analyzing the spinner, predict whether a person would win money or lose money after many spins.

b. Now calculate the actual expected value. How does the result compare to your estimate from part (a)?

c. What would the expected value be if this spinner were fair? Discuss this with your team. What does it mean for a spinner to be fair?

d. How could you change the spinner to make it fair? Draw your new spinner and show why it is fair.

4-104. DOUBLE SPIN



"Double Spin" is a new game. The player gets to spin a spinner twice, but wins only if the same amount comes up both times. The $100 sector is $\frac{1}{8}$ of the circle.

a. Use an area model or tree diagram to show the sample space and probability of each outcome of two spins and then answer the following questions.

b. What is the expected value when playing this game? That is, what is the average amount of money the carnival should expect to pay to players each turn over a long period of time?

c. If it costs $3.00 for you to play this game, should you expect to break even in the long run?

d. Is this game fair?

4-105. BASKETBALL: Shooting One-and-One Free Throws Revisited

Recall the One-and-One situation from problem 4-78. In this problem, Dunkin' Delilah Jones has a 60% free throw average.

a. Use an appropriate model to represent the sample space and then find what would be the most likely result when she shoots a one-and-one.

b. Is it more likely that Delilah would make no points or that she would score some points? Explain.

c. On average, how many points would you expect Dunkin' Delilah to make in a one and one free throw situation? That is, what is the expected value?

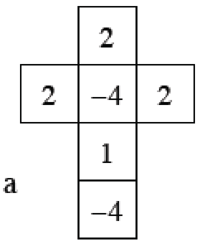
d. Repeat part (a) for at least three other possible free throw percentages, making a note of the most likely outcome for each one.

e. Is there a free throw percentage that would make two points and zero points equally likely outcomes? If so, find this percentage.

f. If you did not already do so, draw an area model or tree diagram for part (e) using x as the percentage and write an equation to represent the problem. Write the solution to the equation in simplest radical form.

4-106. Janine's teacher has presented her with an opportunity to raise her grade: She can roll a special die and possibly gain points. If a positive number is rolled, Janine gains the number of points indicated on the die. However, if a negative roll occurs, then Janine loses that many points.

Janine does not know what to do! The die, formed when the net at right is folded, offers four sides that will increase her number of points and only two sides that will decrease her grade. She needs your help to determine if this die is fair.



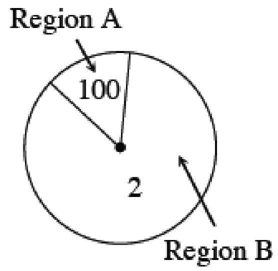
a. What are the qualities of a fair game? How can you tell if a game is fair? Discuss this with your team and be ready to share your ideas with the class.

b. What is the expected value of one roll of this die? Show how you got your answer. Is this die fair?

c. Change only one side of the die in order to make the expected value 0.

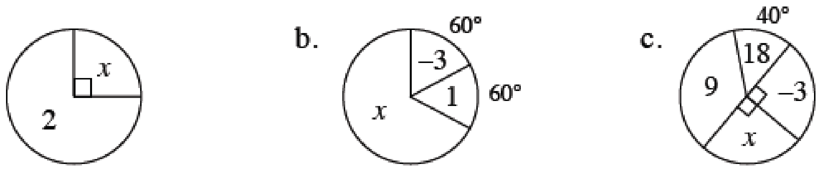
d. What does it mean if a die or spinner has an expected value of 0?

4-107. Examine the spinner at right. If the central angle of Region A is 7°, find the expected value of one spin two different ways. Be ready to share your methods with the class.



4-108. Now reverse the process. For each spinner below, find x so that the expected value of the spinner is 3. Be prepared to explain your method to the class.

a. b. c.



4-109. Revisit your work from part (c) of problem 4-108.

a. To solve for x, Julia wrote the equation:

$$\frac{140}{360}(9)+\frac{40}{360}(18)+\frac{90}{360}(-3)+\frac{90}{360}x=3$$

Explain how her equation works.

b. She is not sure how to solve her equation. She would like to rewrite the equation so that it does not have any fractions. What could she do to both sides of the equation to eliminate the fractions? Rewrite her equation and solve for x.

c. If you have not done so already, write an equation and solve for x for parts (a) and (b) of problem 4-108. Did your answers match those you found in problem 4-108?

Review & Preview

4-110. When he was in first grade, Harvey played games with spinners. One game he especially liked had two spinners and several markers that you moved around a board. You were only allowed to move if your color came up on *both* spinners.

a. Harvey always chose purple because that was his favorite color. What was the probability that Harvey could move his marker?

b. Is the event that Harvey wins a union or an intersection of events?

c. Was purple the best color choice? Explain.

d. If both spinners are spun, what is the probability that no one gets to move because the two colors are not the same?

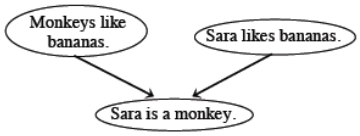
e. There are at least two ways to figure out part (d). Discuss your solution method with your team and show a second way to solve part (d).

4-111. Consider the sequence $2, 8, 3y+5, \ldots$

a. Find the value of y if the sequence is arithmetic.

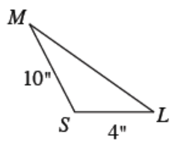
b. Find the value of y if the sequence is geometric.

4-112. What is wrong with the argument shown in the flow chart at right? What assumption does the argument make?

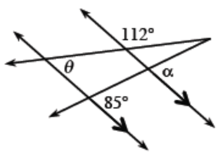


4-113. Kamillah decided to find the height of the Empire State Building. She walked 1 mile away (5280 feet) from the tower and found that she had to look up 15.5° to see the top. Assuming Manhattan is flat, if Kamillah's eyes are 5 feet above the ground how tall is the Empire State Building?

4-114. What are the possible lengths for side $\overline{ML}$ in the triangle at right? Show how you know.



4-115. Find the values of θ and α in the diagram at right. State the relationships you used.



4-116. Avery has been learning to play some new card games and is curious about the probabilities of being dealt different cards from a standard 52-card deck. Help him figure out the probabilities listed below.

a. What are P(king), P(queen), and P(club)?

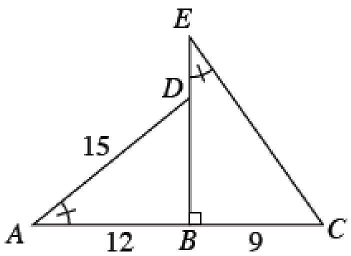
b. What is P(king or club)? How does your answer relate to the probabilities you calculated in part (a)?

c. What is P(king or queen)? Again, how does your answer relate to the probabilities you calculated in part (a)?

d. What is the probability of not getting a face card? Jacks, queens, and kings are face cards.

4-117. Woottonville currently has a population of 1532 people and is growing at a rate of approximately 15 people per year. Nearby, Coynertown has a population of 2740 people but is decreasing at a rate of approximately 32 people per year. In how many years will the towns have the same population?

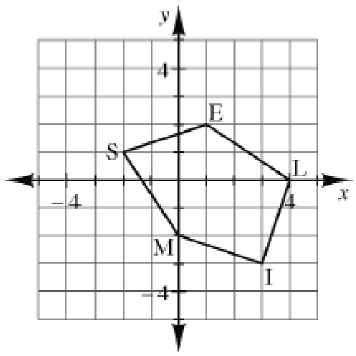
4-118. Examine the diagram at right. If $\overline{AC}$ passes through point B, then answer the questions below.



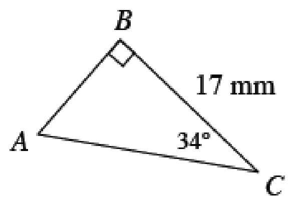
a. Are the triangles similar? If so, make a flowchart justifying your answer.

b. Are the triangles congruent? Explain how you know.

4-119. Examine pentagon *SMILE* at right. Do any of its sides have equal length? How do you know? Be sure to provide convincing evidence. You might want to copy the figure onto graph paper.



4-120. Find the area of the triangle at right. Show all work.



4-121. On graph paper, plot ΔABC if $A(-1,-1)$, $B(3,-1)$, and $C(-1,-2)$.

a. Enlarge (dilate) ΔABC from the origin so that the ratio of the side lengths is 3. Name this new triangle $\Delta A'B'C'$. List the coordinates of $\Delta A'B'C'$.

b. Rotate $\Delta A'B'C'$ 90° clockwise (↻) about the origin to find $\Delta A''B''C''$. List the coordinates of $\Delta A''B''C''$.

c. If ΔABC is translated so that the image of point A is located at $(5, 3)$, where would the image of point B lie?

Game of Chance

2	5	–1	★	–1	5	2

Lesson 4.2.5 Resource Page

Chapter 4 Closure What have I learned?

Reflection and Synthesis

The activities below offer you a chance to reflect about what you have learned during this chapter. As you work, look for concepts that you feel very comfortable with, ideas that you would like to learn more about, and topics you need more help with. Look for connections between ideas as well as connections with material you learned previously.

① TEAM BRAINSTORM

What have you studied in this chapter? What ideas were important in what you learned? With your team, brainstorm a list. Be as detailed as you can. To help get you started, lists of Learning Log entries, Toolkit entries, and Math Notes boxes are below.

What topics, ideas, and words that you learned *before* this chapter are connected to the new ideas in this chapter? Again, be as detailed as you can.

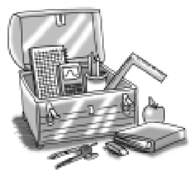
How long can you make your list? Challenge yourselves. Be prepared to share your team's ideas with the class.

Learning Log Entries

- Lesson 4.1.3 – Slope Angles and Slope Ratios
- Lesson 4.1.4 – The Tangent Ratio
- Lesson 4.2.2 – Creating Sample Spaces
- Lesson 4.2.3 – Conditions for Using Probability Methods

Toolkit Entries

- Trig Table Toolkit (Lesson 4.1.2 or 4.1.3 Resource Page and problems 4-16 and 4-25)



Math Notes

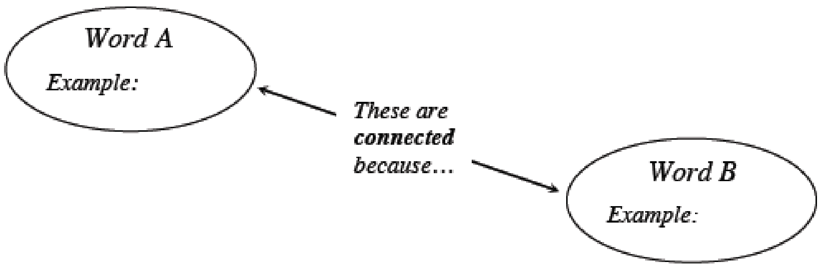
- Lesson 4.1.1 – Slope and Angle Notation
- Lesson 4.1.2 – Slope Ratios and Angles
- Lesson 4.1.3 – Sequences
- Lesson 4.1.4 – Tangent Ratio
- Lesson 4.1.5 – Independent Events
- Lesson 4.2.1 – Solving a Quadratic Equation
- Lesson 4.2.3 – Probability Models
- Lesson 4.2.4 – Intersections, Unions, and Complements

② MAKING CONNECTIONS

Below is a list of the vocabulary used in this chapter. Make sure that you are familiar with all of these words and know what they mean. Refer to the glossary or index for any words that you do not yet understand.

α (alpha)	Addition Rule	angle
arithmetic sequence	clinometer	common difference
common ratio/multiplier	complement	conjecture
equally likely	expected value	fair game
geometric sequence	hypotenuse	independent events
intersection	leg	non-independent events
orientation	probability	probability area model
random	ratio	sample space
slope angle	slope ratio	slope triangle
systematic list	tangent ratio	θ (theta)
tree diagram	trigonometry	union
Δx	Δy	

Make a concept map showing all of the connections you can find among the key words and ideas listed above. To show a connection between two words, draw a line between them and explain the connection, as shown in the model below. A word can be connected to any other word as long as you can justify the connection. For each key word or idea, provide an example or sketch that shows the idea.



Your teacher may provide you with vocabulary cards to help you get started. If you use the cards to plan your concept map, be sure either to re-draw your concept map on your paper or to glue the vocabulary cards to a poster with all of the connections explained for others to see and understand.

While you are making your map, your team may think of related words or ideas that are not listed here. Be sure to include these ideas on your concept map.

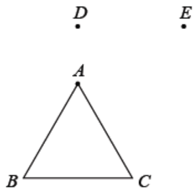
③ PORTFOLIO: EVIDENCE OF MATHEMATICAL PROFICIENCY

Visualization is required when you imagine a situation and want to draw a diagram to represent it. Read the descriptions below and visualize what each situation looks like. Then draw a diagram for each. Label your diagrams appropriately with any given measurements.

a. Karen is flying a kite on a windy day. Her kite is 80 feet above ground and her string is 100 feet long. Karen is holding the kite 3 feet above ground.

b. The bow of a rowboat (which is 1 foot above water level) is tied to a point on a dock that is 6 feet above the water level. The length of the rope between the dock and the boat is 8 feet.

Sometimes, visualization requires you to think about how an object can move in relation to others. For example, consider equilateral ΔABC at right.

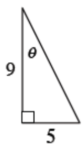


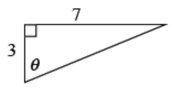
c. Visualize changing ΔABC by stretching vertex A to point D, which is directly above point A. What does the new triangle look like? Do you have a name for it?

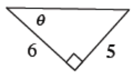
d. What happened to $m\angle A$ as you stretched the triangle in part (1)? What happened to $m\angle B$ and $m\angle C$?

e. Now visualize the result after vertex A stretched to point E. What type of triangle is the result? What happens to $m\angle B$ as the triangle is stretched? What happens to $m\angle C$?

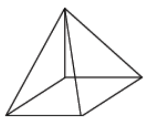
An important use of visualization is to re-orient a right triangle to help you identify which leg is Δx and which leg is Δy. For each triangle below, visualize the triangle by rotating and/or reflecting it so that it is a slope triangle. Draw the result and label the appropriate legs Δx or Δy.

f.


g.


h.


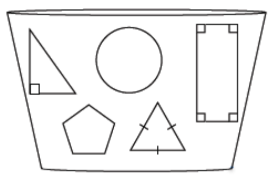
Finally, visualization can help you view an object from different perspectives. For example, consider the square-based pyramid at right. Visualize what you would see if you looked down at the pyramid from a point directly above the top vertex. Draw this view on your paper.



Next, showcase your understanding of probability by solving these problems. Explain your thinking in detail.

i. Harold sorted his jellybeans into two jars. He likes purple ones best and the black ones next best, so they are both in one jar. His next favorites are yellow, orange, and white, and they are in another jar. He gave all the rest to his little sister. Harold allows himself to eat only one jellybean from each jar per day. He wears a blindfold when he selects his jellybeans so he cannot choose his favorites first. Show a complete sample space. What is the probability that Harold gets one black jellybean and one orange jellybean, if the first jar has 60% black and 40% purple jellybeans and the second jar has 30% yellow, 50% orange, and 20% white jellybeans?

j. A game is set up so that a person randomly selects a shape from the shape bucket shown at right. If the person selects a triangle, he or she wins \$5. If the person selects a circle, he or she loses \$3. If any other shape is selected, the person does not win or lose money. If a person plays 100 times, how much money should the person expect to win or lose? If you play this game many times, what can you expect to win (what is the expected value)? Is this game fair?



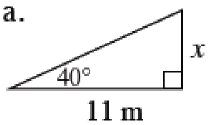
Your teacher may give you the Chapter 4 Closure Resource Page: Tangent Graphics Organizer to work on (or you can download this page from www.cpm.org). A Graphic Organizer is a tool you can use to organize your thoughts, showcase your knowledge, and communicate your ideas clearly.

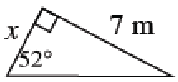
④ WHAT HAVE I LEARNED?

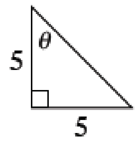
Most of the problems in this section represent typical problems found in this chapter. They serve as a gauge for you. You can use them to determine which types of problems you can do well and which types of problems require further study and practice. Even if your teacher does not assign this section, it is a good idea to try these problems and find out for yourself what you know and what you still need to work on.

Solve each problem as completely as you can. The table at the end of the closure section has answers to these problems. It also tells you where you can find additional help and practice with problems like these.

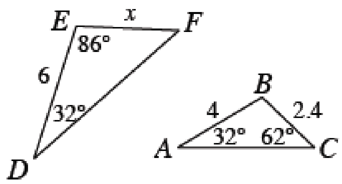
CL 4-122. Solve for the missing side length or angle below.

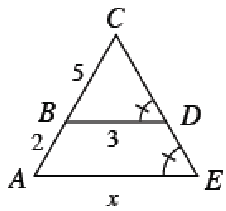
a.


b.


c.


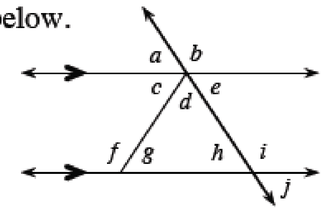
CL 4-123. Use a flowchart to show how you know the triangles are similar. Then find the value of each variable.

a.


b.


CL 4-124. Salvador has a hot dog stand 58 meters from the base of the Space Needle in Seattle. He prefers to work in the shade and knows that he can calculate when his hotdog stand will be in the shade if he knows the height of the Space Needle. To measure its height, Salvador stands at the hotdog stand, gets out his clinometer, and measures the angle to the top of the Space Needle to be 80°. Salvador's eyes are 1.5 meters above the ground. Assuming that the ground is level between the hotdog stand and the Space Needle, how tall is the Space Needle?

CL 4-125. Use the diagram at right to answer the questions below.



a. State the name of the geometric relationship between the angles below. Also describe the relationship between the angle measures, if one exists.

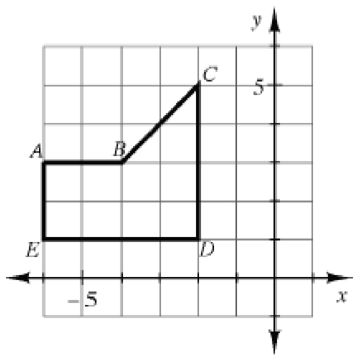
i. $\angle a$ and $\angle h$ *ii.* $\angle b$ and $\angle e$

iii. $\angle c$ and $\angle g$ *iv.* $\angle g$, $\angle d$, and $\angle h$

b. Find the measure of each angle listed below and justify your answer. Let $m\angle c = 32°$ and $m\angle e = 55°$ in the figure above.

i. $m\angle j$ *ii.* $m\angle d$ *iii.* $m\angle a$ *iv.* $m\angle g$

CL 4-126. Draw a pair of axes in the center of a half sheet of graph paper. Then draw the figure at right and perform the indicated transformations. For each transformation, label the resulting image $A'B'C'D'E'$.



a. Rotate *ABCDE* 180° ↻ around the origin.

b. Rotate *ABCDE* 90°↺ around the origin.

c. Reflect *ABCDE* across the *y*-axis.

d. Translate *ABCDE* up 5, left 7.

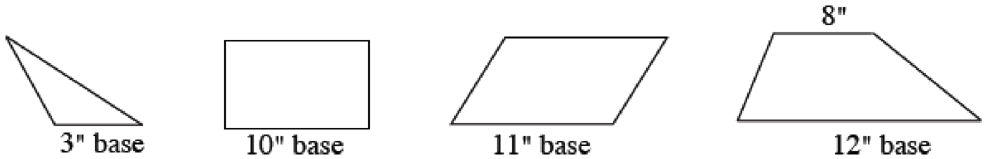
CL 4-127. Kiyomi has 4 pairs of pants (black, peach, gray, and cream), and she has 5 shirts (white, red, teal, black, and lavender).

a. If any shirt can be worn with any pair of pants, represent the sample space of all possible outfits with both a probability area model and a tree diagram. How many outfits does she own?

b. The closet light is burned out, so Kiyomi must randomly select a pair of pants and a shirt. What is the probability that she will wear something black?

CL 4-128. In a certain town, 45% of the population has dimples and 70% has a widow's peak (a condition where the hairline above the forehead makes a "V" shape). Assuming that these physical traits are independently distributed, what is the probability that a randomly selected person has both dimples and a widow's peak? What is the probability that he or she will have neither? Use a probability area model or a tree diagram to represent this situation.

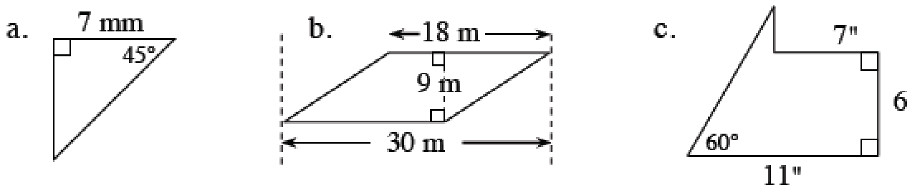
CL 4-129. Trace each figure onto your paper and label the sides with the given measurements.



a. On your paper, draw a height that corresponds to the labeled base for each figure.

b. Assume that the height for each figure above is 7 inches. Add this information to your diagrams and find the area of each figure.

CL 4-130. Find the perimeter of each shape below. Assume the diagram in part (b) is a parallelogram.



CL 4-131. For each equation below, solve for x.

a. $\frac{x}{23}=\frac{15}{7}$

b. $(x+2)(x-5)=6x+x^2-5$

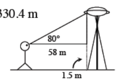
c. $x^2+2x-15=0$

d. $2x^2-11x=-3$

CL 4-132. Check your answers using the table at the end of this section. Which problems do you feel confident about? Which problems were hard? Have you worked on problems like these in math classes you have taken before? Use the table to make a list of topics you need help on and a list of topics you need to practice more.

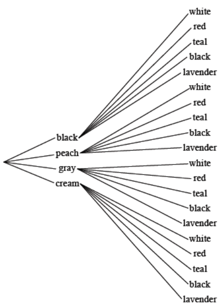
Answers and Support for Closure Activity #4
What Have I Learned?

MN = Math Notes, LL = Learning Logs

Problem	Solution	Need Help?	More Practice
CL 4-122.	a. $x \approx 9.23$ b. $x \approx 5.47$ c. $\theta = 45°$	Section 4.1 MN: 4.1.2 and 4.1.4 LL: 4.1.3 and 4.1.4	Problems 4-39, 4-63, and 4-74
CL 4-123.	a. $x \approx 3.6$ $m\angle EDF = m\angle BAC$, $m\angle DEF = m\angle ABC$ → $\Delta EDF \sim \Delta BAC$ b. $x = 4.2$ $m\angle EDF = m\angle BAC$, $m\angle BCD = m\angle ACE$ → $\Delta BCD \sim \Delta ACE$	Section 3.2 MN: 3.2.1, 3.2.4, and 3.2.5 LL: 3.2.2 and 3.2.4	Problems CL 3-121, 4-7, 4-41, 4-70, 4-42, and 4-118
CL 4-124.	Total height ≈ 330.4 m  not to scale	Lesson 4.1.5 MN: 4.1.4 LL: 4.1.4	Problems 4-43, 4-50, 4-83, and 4-113
CL 4-125.	a. *i*. corresponding angles, congruent *ii*. straight angle pair, supplementary *iii*. alternate interior angles, congruent *iv*. triangle angle sum is 180° b. *i*. 55°: corresponding to $\angle e$ *ii*. 93°: straight angle with $\angle e$ and $\angle c$ *iii*. 55°: vertical to $\angle e$ *iv*. 32°: alternate interior to $\angle c$	Section 2.1 MN: 2.1.1, 2.1.4, and 2.2.1 LL: 2.1.1	Problems CL 2-122, 4-48, and 4-115
CL 4-126.	a. $A'(6,-3)$, $B'(4,-3)$, $C'(2,-5)$, $D'(2,-1)$, $E'(6,-1)$ b. $A'(-3,-6)$, $B'(-3,-4)$, $C'(-5,-2)$, $D'(-1,-2)$, $E'(-1,-6)$ c. $A'(6,3)$, $B'(4,3)$, $C'(2,5)$, $D'(2,1)$, $E'(6,1)$ d. $A'(-13,8)$, $B'(-11,8)$, $C'(-9,10)$, $D'(-9,6)$, $E'(-13,6)$	Lessons 1.2.2 and 1.2.4 MN: 1.2.2 and 1.2.4	Problems CL 1-128, CL 2-124, 3-76, and 4-121
CL 4-127.	a. See diagrams below. 20 outfits b. 8 of the outcomes in the sample space contain a black item out of 20 possible outfits. $\frac{8}{20} = \frac{2}{5}$	Lessons 4.2.1, 4.2.2, and 4.2.3 MN: 4.2.3 LL: 4.2.2 and 4.2.3	Problems 4-69, 4-81, 4-95, and 4-96

shirts

pants	white	red	teal	black	lavender
black	white shirt, black pants	red shirt, black pants	teal shirt, black pants	black shirt, black pants	lavender shirt, black pants
peach	white shirt, peach pants	red shirt, peach pants	teal shirt, peach pants	black shirt, peach pants	lavender shirt, peach pants
gray	white shirt, gray pants	red shirt, gray pants	teal shirt, gray pants	black shirt, gray pants	lavender shirt, gray pants
cream	white shirt, cream pants	red shirt, cream pants	teal shirt, cream pants	black shirt, cream pants	lavender shirt, cream pants



Problem	Solution	Need Help?	More Practice
CL 4-128.	P(both) = 31.5% P(neither) = 16.5%	Section 4.2 MN: 4.2.3 and 4.2.4 LL: 4.2.3	Problems 4-69, 4-81, 4-95, and 4-96

Dimples?

Widow's Peak?	Yes $\frac{45}{100}$	No $\frac{55}{100}$
Yes $\frac{70}{100}$	31.5%	38.5%
No $\frac{30}{100}$	13.5%	16.5%

Problem	Solution	Need Help?	More Practice
CL 4-129.	a. All heights should be 7 inches. b. 10.5", 70", 77", 70"	Section 2.2 MN: 1.1.3 and 2.2.4 Area Toolkit	Problems 4-20, 4-44, and 4-120
CL 4-130.	a. ≈ 23.899 mm b. = 66 m c. ≈ 32.93 inches	Lesson 4.1.3 MN: 1.1.3 and 4.1.4 LL: 2.2.2	Problems 4-20, 4-39, 4-44, 4-63, and 4-74
CL 4-131.	a. $x = 49.286$ b. $x = -\frac{5}{9}$ c. $x = 3$ or $x = -5$ d. $x = \frac{11 \pm \sqrt{97}}{4} \approx 5.21$ or 0.29	MN: 1.1.4, 2.2.2, and 4.2.1	Problems CL 1-133, CL 3-120, 4-62, and 4-84

Chapter 4 Closure Resource Page: Concept Map Cards Page 1 of 4

α (alpha)	Addition Rule
Angle	Arithmetic sequence
Clinometer	Common difference
Common ratio/multiplier	Complement
Conjecture	Equally likely

Expected value	Fair game
Geometric sequence	Hypotenuse
Independent events	Intersection
Leg	Non-independent events
Orientation	Probability

Probability area model	Random
Ratio	Sample space
Slope angle	Slope ratio
Slope triangle	Systematic list
Tangent ratio	θ (theta)

Tree diagram	Trigonometry
Union	Δx
Δy	

Chapter 4 Closure Resource Page: Tangent Graphic Organizer

Tangent Graphic Organizer

For each pair of terms below, write everything you know about the relationship between them. Be sure to include diagrams