2-4 Study Guide and Intervention

Zeros of Polynomial Functions

Finding Real Zeros of Polynomial Functions The **Rational Zero Theorem** describes how the leading coefficient and constant term of a polynomial function with integer coefficients determine a list of all possible rational zeros.

$$\text{Possible Rational Zeros} = \frac{\text{Factors of the constant}}{\text{Factors of the leading coefficient}}$$

Descartes' Rule of Signs states that the number of positive real zeros is equal to the number of sign changes in $f(x)$ or less than that by an even number. The number of negative real zeros is the same as the number of sign changes in $f(-x)$ or less than that by an even number.

Example

a. List all possible rational zeros of $h(x) = x^3 - 5x^2 - 17x - 6$. Then determine which, if any, are zeros.

The leading coefficient is **1** and the constant term is **–6**.

Possible rational zeros: $\frac{\text{Factors of } -6}{\text{Factors of } 1} = \frac{\pm 1,\ \pm 2,\ \pm 3,\ \pm 6}{\pm 1}$ or $\pm 1, \pm 2, \pm 3, \pm 6$

By synthetic substitution, you can determine that $x = -2$ is a rational zero.

–2	1	–5	–17	–6
		–2	14	6
	1	–7	–3	0

The depressed polynomial is $x^2 - 7x - 3$. You can use the quadratic formula to find the two irrational roots.

b. Describe the possible real zeros of $g(x) = 4x^4 - 13x^3 - 21x^2 + 38x - 8$.

Examine the variations in sign for $g(x)$ and for $g(-x)$.

– to +

$$g(x) = 4x^4 - 13x^3 - 21x^2 + 38x - 8$$

+ to – + to –

$$g(-x) = 4(-x)^4 - 13(-x)^3 - 21(-x)^2 + 38(-x) - 8$$

$$= 4x^4 + 13x^3 - 21x^2 - 38x - 8$$

+ to –

By Descartes' Rule of Signs, $g(x)$ has either 3 or 1 positive real zeros and 1 negative real zero. The graph shows it has 1 negative real zero and 3 positive real zeros.

Exercises

1. List all possible rational zeros of $h(x) = x^3 - 5x^2 - 4x + 20$. Then determine which, if any, are zeros.

$$\frac{\text{Factors of } 20}{\text{Factors of } 1} = \frac{\pm 1, \pm 2, \pm 4, \pm 5, \pm 10, \pm 20}{\pm 1}$$

synthetically divide $= \pm 1, \pm 2, \pm 4, \pm 5, \pm 10, \pm 20$

-2, 2, and 5

2. Describe the possible real zeros of $f(x) = x^5 - 10x^4 + 7x^3 - x^2 + 4x + 20$.

pos. or neg.

positive zeros

$f(x) = x^5 - 10x^4 + 7x^3 - x^2 + 4x + 20$

\+ to −   − to +   + to −   − to +

4 sign changes means 4 positive real zeros or 2 positive real zeros.

negative zeros

$f(-x) = (-x)^5 - 10(-x)^4 + 7(-x)^3 - (-x)^2 + 4(-x) + 20$

$= -x^5 - 10x^4 - 7x^3 - x^2 - 4x + 20$

− to +

There is 1 negative real zero

Zeros of Polynomial Functions

Finding Complex Zeros of Polynomial Functions Polynomials can have zeros in the complex number system. The **Fundamental Theorem of Algebra** states that a polynomial function of degree n has at least one zero in the complex number system. In fact, it has exactly n zeros, including repeated zeros. The **Conjugate Root Theorem** states that if $a + bi$ is a zero, then its complex conjugate $a - bi$ is also a zero.

Example 1: **Write a polynomial function of least degree with real coefficients in standard form that has 0 and $\sqrt{2}i$ as zeros.**

Because $\sqrt{2}i$ is a zero, you know that $-\sqrt{2}i$ is also a zero.

$f(x) = a(x - 0)(x - \sqrt{2}i)\,[x - (-\sqrt{2}i)]$	Write the function as factors.
$f(x) = 1x(x - \sqrt{2}i)(x + \sqrt{2}i)$	Let a = 1. Simplify.
$= x(x^2 - 2i^2)$	Multiply.
$= x(x^2 + 2)$	Simplify. $i^2 = -1$
$= x^3 + 2x$	Distribute.

A function of least degree that has 0, $\sqrt{2}i$, and $-\sqrt{2}i$ as zeros is $f(x) = x^3 + 2x$.

Example 2: **Find all complex zeros of $p(x) = x^4 - 5x^3 + 3x^2 + 19x - 30$ given that $x = 2 + i$ is a zero of p. Then write the linear factorization of $p(x)$.**

Because $x = 2 + i$ is a zero, you know that $x = 2 - i$ is also a zero of p. First divide by $2 + i$ to get the depressed polynomial. Then divide it by $2 - i$.

$2+i$	1	−5	3	19	−30
		$2+i$	$-7-i$	$-7-6i$	30
	1	$-3+i$	$-4-i$	$12-6i$	0

$2-i$	1	$-3+i$	$-4-i$	$12-6i$
		$2-i$	$-2+i$	$-12+6i$
	1	−1	−6	0

Using the known zeros and the depressed polynomial from the last division, you can write $p(x) = [x - (2 + i)]\,[x - (2 - i)]\,(x^2 - x - 6)$.

Factor the remaining depressed polynomial, $x^2 - x - 6 = (x - 3)(x + 2)$.

The four zeros of p are $2 + i$, $2 - i$, 3, and −2. The linear factorization of p is $p(x) = [x - (2 + i)]\,[x - (2 - i)]\,(x - 3)(x + 2)$.

Exercises

1. Write a polynomial function of least degree with real coefficients in standard form that has the zeros –4, 1, and 4 + *i*.

$(x+4)(x-1)(x-(4+i))(x-(4-i))$

$x^2-x+4x-4 \qquad x^2+4x+xi+4x+xi+16-i^2$

$=(x^2+3x-4)(x^2-8x+17)$

$=x^4-8x^3+17x^2+3x^3-24x^2+51x-4x^2+32x-68$

$=x^4-5x^3-11x^2+83x-68$

2. Find all complex zeros of $p(x) = x^4 + 8x^3 + 16x^2 + 200x - 225$ given that $x = 5i$ is a zero of p. Then write the linear factorization of $p(x)$.

complex zeros: $5i$ & $-5i$

$5i$	1	8	16	200	-225
	↓	$5i$	$-25+40i$	$-200-45i$	225
	1	$8+5i$	$-9+40i$	$-45i$	0

$-5i$	1	$8+5i$	$-9+40i$	$-45i$
	↓	$-5i$	$-40i$	$45i$
	1	8	-9	0

x^2+8x-9

$p(x)=(x-1)(x+9)(x+5i)(x-5i)$

HW: p.127, # 1,7,11,15,19,27,29,33,39,43,49