6-4 Partial Fractions

Then

You graphed rational functions. (Lesson 2-4)

Now

- Write partial fraction decompositions of rational expressions with linear factors in the denominator.
- Write partial fraction decompositions of rational expressions with prime quadratic factors in the denominator.

7. a)

$$\begin{cases} d + b + c = 25 \\ 4d + 7b + 3c = 100 \\ b + 10 = d \end{cases}$$

$$\begin{aligned} b + 10 &= d \\ -b \quad & \quad -b \\ d - b &= 10 \end{aligned}$$

b)

$$\text{rref}\left[\begin{array}{ccc|c} 1 & 1 & 1 & 25 \\ 4 & 7 & 3 & 100 \\ 1 & -1 & 0 & 10 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 13 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 9 \end{array}\right]$$

$d = 13$ lbs of dog food
$b = 3$ lbs of bird seed
$c = 9$ lbs of cat food

0-2 +
3 ½ ✓
4↑ −

6-4 Study Guide and Intervention

Partial Fractions

Linear Factors The function $g(x)$ shown below can be written as the sum of two fractions with denominators that are linear factors of the original denominator.

$$g(x) = \frac{3x-1}{2x^2-3x+1} = \boxed{\frac{2}{x-1} + \frac{-1}{2x-1}}$$

Each fraction in the sum is a **partial fraction.** The sum of these partial fractions make up the **partial fraction decomposition** of the original rational function. If the denominator of a rational expression contains a repeated linear factor, the partial fraction decomposition must include a partial fraction with its own constant numerator for each power of this factor.

Example : Find the partial fraction decomposition of $\frac{x+11}{x^2-3x-4}$.

Rewrite the equation as partial fractions with constant numerators, A and B, and denominators that are the linear factors of the original denominator.

$\frac{x+11}{x^2-3x-4} = \frac{A}{x-4} + \frac{B}{x+1}$	Form a partial fraction decomposition.
$x + 11 = A(x+1) + B(x-4)$	Multiply each side by the LCD, $x^2 - 3x - 4$.
$x + 11 = Ax + A + Bx - 4B$	Distributive Property
$1x + 11 = (A+B)x + (A + (-4B))$	Group like terms.

Equate the coefficients on the left and right side of the equation to obtain a system of two equations. To solve the system, write it in matrix form $CX = D$ and solve for X.

$$\begin{aligned} A + B &= 1 \\ A + (-4B) &= 11 \end{aligned} \quad \longrightarrow \quad \underset{C}{\begin{bmatrix} 1 & 1 \\ 1 & -4 \end{bmatrix}} \cdot \underset{X}{\begin{bmatrix} A \\ B \end{bmatrix}} = \underset{D}{\begin{bmatrix} 1 \\ 11 \end{bmatrix}}$$

Solving for X yields $A = 3$ and $B = -2$. Therefore $\frac{x+11}{x^2-3x-4} = \frac{3}{x-4} + \frac{-2}{x+1}$.

Find the partial fraction decomposition of the rational expression $\frac{7x}{x^2+3x-10}$.

$$\frac{7x}{\cancel{x^2+3x-10}}(\cancel{(x-2)(x+5)}) = \frac{A}{\cancel{x-2}}(\cancel{(x-2)}(x+5)) + \frac{B}{\cancel{x+5}}((x-2)\cancel{(x+5)})$$

$$7x = A(x+5) + B(x-2)$$

$$7x + 0 = Ax + 5A + Bx - 2B$$

$$7x + 0 = (Ax + Bx) + (5A - 2B)$$

$$7x + 0 = (A+B)x + (5A-2B)$$

$$\begin{cases} A + B = 7 \\ 5A - 2B = 0 \end{cases}$$

$$\begin{bmatrix} 1 & 1 \\ 5 & -2 \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 7 \\ 0 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \frac{2}{7} & \frac{1}{7} \\ \frac{5}{7} & \frac{-1}{7} \end{bmatrix}$$

$$A^{-1}B = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

$$\boxed{\frac{2}{x-2} + \frac{5}{x+5}}$$

$A = 2 \quad B = 5$

Exercises

Find the partial fraction decomposition of each rational expression.

1. $\frac{5x-34}{x^2-x-12} = \frac{A}{x-4} + \frac{B}{x+3}$

$5x-34 = A(x+3) + B(x-4)$

$5x-34 = Ax + 3A + Bx - 4B$

$5x-34 = (Ax+Bx) + (3A-4B)$

$5x-34 = (A+B)x + (3A-4B)$

$\begin{cases} A+B=5 \\ 3A-4B=-34 \end{cases}$

$\begin{bmatrix} 1 & 1 \\ 3 & -4 \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 5 \\ -34 \end{bmatrix}$

$A^{-1} = \begin{bmatrix} \frac{4}{7} & \frac{1}{7} \\ \frac{3}{7} & -\frac{1}{7} \end{bmatrix}$

$A^{-1}B = \begin{bmatrix} -2 \\ 7 \end{bmatrix}$

$A=-2 \quad B=7$

$\frac{-2}{x-4} + \frac{7}{x+3}$

2. $\frac{-7x+13}{x^2-5x-14} = \frac{A}{x-7} + \frac{B}{x+2}$

$-7x+13 = A(x+2) + B(x-7)$

$-7x+13 = Ax + 2A + Bx - 7B$

$-7x+13 = (A+B)x + (2A-7B)$

$\begin{cases} A+B=-7 \\ 2A-7B=13 \end{cases}$

$\begin{bmatrix} 1 & 1 \\ 2 & -7 \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} -7 \\ 13 \end{bmatrix}$

$A^{-1} = \begin{bmatrix} \frac{7}{9} & \frac{1}{9} \\ \frac{2}{9} & -\frac{1}{9} \end{bmatrix}$

$A^{-1}B = \begin{bmatrix} -4 \\ -3 \end{bmatrix}$

$A=-4 \quad B=-3$

$\frac{-4}{x-7} + \frac{-3}{x+2}$

HW: p. 402
1-6, 57, 61, 64-67

57.

$$\begin{cases} a-2b+c=7 \\ 6a+2b-2c=4 \\ 4a+6b+4c=14 \end{cases}$$

rref $\left[\begin{array}{ccc|c} 1 & -2 & 1 & 7 \\ 6 & 2 & -2 & 4 \\ 4 & 6 & 4 & 14 \end{array}\right]$

$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right]$ $a=2$, $b=-1$, $c=3$ $(2,-1,3)$

0-3 +
4-5 ✓
6 ↑ -

EXAMPLE 2 Improper Rational Expression

Find the partial fraction decomposition of $\dfrac{-2x^2+9x-4}{x^2-x}$.

Because the degree of the numerator is greater than or equal to the degree of the denominator, the rational expression is improper. To rewrite the expression, divide the numerator by the denominator using polynomial division.

EXAMPLE 2 Improper Rational Expression

$$
\begin{array}{r}
-2 \\
x^2 - x \overline{\smash{\big)}\,-2x^2 + 9x - 4} \\
(-)\underline{(-2x^2 + 2x)} \\
7x - 4
\end{array}
$$

← Multiply the divisor by -2 because $\frac{-2x^2}{x^2} = -2$.

← Subtract and bring down next term.

So, the original expression is equal to $-2 + \frac{7x-4}{x^2 - x}$.

Because the remaining rational expression is now proper, you can factor its denominator as $x(x - 1)$ and rewrite the expression using partial fractions.

EXAMPLE 2 Improper Rational Expression

$\dfrac{7x-4}{x^2-x}=\dfrac{A}{x}+\dfrac{B}{x-1}$	Form of decomposition
$7x-4=A(x-1)+Bx$	Multiply by the LCD, x^2-x.
$7x-4=Ax-A+Bx$	Distributive Property
$7x-4=(A+B)X-A$	Group like terms.

Write and solve the system of equations obtained by equating the coefficients.

$$\begin{aligned} A+B &= 7 \\ -A &= -4 \end{aligned} \quad \rightarrow \quad \begin{aligned} A &= 4 \\ B &= 3 \end{aligned}$$

Therefore, $\dfrac{-2x^2+9x-4}{x^2-x}=-2+\dfrac{7x-4}{x^2-x}$ or $-2+\dfrac{4}{x}+\dfrac{3}{x-1}$.

Find the partial fraction decomposition of

$$\frac{3x^2+7x}{x^2+7x+12}.$$

$$3 + \frac{-14x-36}{x^2+7x+12}$$

$$x^2+7x+12 \overline{\smash{)}\,3x^2+7x+0}$$
$$-(3x^2+21x+36)$$
$$-14x-36$$

$$\frac{-14x-36}{x^2+7x+12} = \frac{A}{x+4} + \frac{B}{x+3}$$

$$-14x-36 = A(x+3) + B(x+4)$$
$$-14x-36 = Ax+3A + Bx+4B$$
$$-14x-36 = (Ax+Bx) + (3A+4B)$$
$$-14x-36 = (A+B)x + (3A+4B)$$

$$\begin{cases} A+B=-14 \\ 3A+4B=-36 \end{cases}$$

$$\begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} -14 \\ -36 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 4 & -1 \\ -3 & 1 \end{bmatrix}$$

$$A^{-1}B = \begin{bmatrix} -20 \\ 6 \end{bmatrix} \quad A=-20 \quad B=6$$

$$\boxed{\frac{-20}{x+4} + \frac{6}{x+3} + 3}$$

HW: p.402 7-12
p.414 23, 25, 31, 33, 35, 43, 45, 47